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ON CERTAIN INVARIANTS INTRODUCED BY REIDEMEISTER*

By J. H. C. WHITEHEAD (*Oxford*)

[Received 2 July 1938]

LET K be any finite, simplicial complex, let U be its universal covering complex and G the group of covering transformations (*Deckbewegungen*) of U into itself. Let $\mathfrak{R}$ be the integral group-ring of G, the elements of $\mathfrak{R}$ being linear forms $n_1 g_1 + n_2 g_2 + \ldots$, in which $g_1, g_2, \ldots$ are elements of G and $n_1, n_2, \ldots$ integers, all but a finite number of which are zero. As Reidemeister explains, the k-chains in U, with integral coefficients, can be written in the form

$$r_1 A_1^k + r_2 A_2^k + \ldots,$$

where $r_1, r_2, \ldots$ are elements in $\mathfrak{R}$ and $A_1^k, A_2^k, \ldots$ are a fundamental set of k-simplexes in U, in the sense that precisely one of them covers each k-simplex in K. The boundary relations may then be written as

$$\dot{A}_i^k = \sum_j r_{ij}^k A_j^{k-1},$$

where $\mathbf{r}^k = \|r_{ij}^k\|$ $(k = 1, 2, \ldots)$ are matrices whose elements belong to $\mathfrak{R}$. Any invariant of these matrices under the following transformations is a combinatorial invariant of K:

A_1: a change of basis of the form

$$C_h^k = \lambda A_h^k + r A_i^k, \qquad C_j^k = A_j^k \qquad (i, j \neq h),$$

where $\lambda = \pm g$ $(g \in G)$ and $r \in \mathfrak{R}$;

A_2: the addition of new chains C_0^k and C_0^{k-1}, where $\dot{C}_0^k$ consists of $\pm g C_0^{k-1}$ together with one of the existing chains;†

A_3: the inverse of A_2.

With the help of these invariants Reidemeister has, among other things, completed the combinatorial classification of lens spaces.

* K. Reidemeister, *Abh. Math. Sem. Hamburg*, 10 (1934), 211–15. For the results concerning lens spaces see K. Reidemeister, ibid. 11 (1935), 102–9, and *J. für Math.* 173 (1935), 164–73, § 6.

† The effect of A_2 is to augment $\mathbf{r}^k$ by a new row, and a new column whose only non-zero element is $\pm g$, which occurs in the new row, and to augment $\mathbf{r}^{k-1}$ by a new row, corresponding to $\dot{C}_0^{k-1}$, which is linearly dependent on the existing rows, since $\dot{C}_0^{k-1} = -(\dot{C}_0^k - C_0^{k-1})$: I work with these, rather than the simpler elementary transformations in which $C_0^{k-1} = \dot{C}_0^k$, in order to simplify the wording later on.

That is to say, he has proved that, if a lens space of type* (p,q) is combinatorially equivalent to one of type (p,q'), then $qq' \equiv \pm 1 \pmod{p}$.

The purpose of this note is to show that Reidemeister's invariants are not merely combinatorial, but also topological invariants. Indeed they are what may be called homotopy invariants, meaning that they are the same for all complexes of the same homotopy type.† This is an obvious consequence of a theorem I have proved elsewhere,‡ according to which two (finite) complexes of the same homotopy type can be interchanged by finite sequences of certain elementary transformations. These depend on a parameter m, which may be arbitrarily chosen in advance, and are of the forms§

$$B_1: \qquad K \to K + aA^p,$$

where aA^p is a $(p+1)$-simplex, with its boundary, such that $a\dot{A}^p \subset K$, $A^p \not\subset K$ (whence $aA^p \not\subset K$) and p is unrestricted;

$$B_2: \qquad K \to K + A^p,$$

where $\dot{A}^p \subset K$, $A^p \not\subset K$, and $p > m$;

B_3: the inverse of B_1 or B_2.

Since $a\dot{A}^p$ is simply-connected, it is covered, if it is in K, by p-elements $a_i \dot{A}^p_i$ $(i = 1,\ldots)$ in U, and $A^p_i \not\subset U$ since $A^p \not\subset K$. Similarly, if $p > 2$ and $\dot{A}^p \subset K$, $A^p \not\subset K$, then $\dot{A}^p$ is covered by $(p-1)$-spheres $\dot{A}^p_i$ in U and $A^p_i \not\subset U$. It follows that the transformation B_1 gives rise to A_2, with $k = p+1$ and $C^k_0 = a_1 A^p_1 \subset U + a_1 A^p_1 + \ldots$, $C^{k-1}_0 = A^p_1$. Also, provided that $p > 2$, B_2 and its inverse have no effect on $\mathbf{r}^1,\ldots,\mathbf{r}^{p-1}$. If fundamental sets of simplexes are taken as bases for the p-chains and $(p+1)$-chains in U, the inverse of B_1 obviously gives rise to the inverse of A_2.

Now let K_1 and K_2 be any two complexes of the same homotopy type, each being at most n-dimensional, where $n \geqslant 2$. Let U_α be the

* H. Seifert and W. Threlfall, *Lehrbuch der Topologie*, Leipzig (1934), 210 and 215.

† W. Hurewicz, *Kon. Akad. Amsterdam*, 39 (1936), 124. We recall that two topological spaces, P and Q, are of the same homotopy type, if there are continuous maps $f(P) \subset Q$ and $g(Q) \subset P$ such that each of the maps $gf(P) \subset P$ and $fg(Q) \subset Q$ is homotopic to the identity.

‡ *Proc. London Math. Soc.* 45 (1939), 243– .

§ Conversely, two complexes of at most n dimensions are of the same homotopy type if they are equivalent under the transformations B with $m = n+1$.

universal covering complex of K_α ($\alpha = 1, 2$) and let $\mathbf{r}_\alpha^1, \mathbf{r}_\alpha^2, \ldots$ be incidence matrices for U_α, taking fundamental sets of simplexes as bases for the chains. If $k > \dim(K_\alpha)$, we define $\mathbf{r}_\alpha^k$ as the 'empty matrix', having no rows or columns, which may be augmented by a transformation of the type A_2. Taking $m = n$, it follows from the last paragraph that, as K_1 is transformed into K_2 by the transformations B, so $\mathbf{r}_1^k$ $(1 \leqslant k \leqslant n)$ is transformed into $\tilde{\mathbf{r}}_2^k$ by transformations of the types A_2 and A_3, where $\tilde{\mathbf{r}}_2^1, \ldots, \tilde{\mathbf{r}}_2^n$ and $\mathbf{r}_2^1, \ldots, \mathbf{r}_2^n$ differ only in the choice of fundamental sets of simplexes in U_2. The matrices $\tilde{\mathbf{r}}_2^1, \ldots, \tilde{\mathbf{r}}_2^n$ and $\mathbf{r}_2^1, \ldots, \mathbf{r}_2^n$ are therefore equivalent under transformations of the type A_1, and Reidemeister's invariants are seen to be the same for K_1 and K_2. Hence, if two lens spaces, of type (p, q) and (p, q') respectively, are of the same homotopy type, then $qq' \equiv \pm 1 \pmod{p}$ and it follows that they are combinatorially equivalent.

ON A SET OF LINEAR EQUATIONS (II)

E. H. LINFOOT (*Bristol*) and W. M. SHEPHERD (*Bristol*)

[Received 3 October 1938]

1. In a previous note we showed† that the equations

$$\sum_{q=0}^{\infty} \frac{\alpha_q}{\lambda - r + q} = 0 \quad (r = 0, 1, 2, \ldots)$$

(where λ is not an integer) in the infinitely many unknowns $\alpha_0, \alpha_1, \ldots$ are only consistent if $\lambda > 0$ with every $\alpha_q = 0$, while, if $\lambda < 0$ and $p = [-\lambda]$, their most general solution is

$$\alpha_q = c_0 \binom{\lambda+q}{q} + c_1 \binom{\lambda+q+1}{q} + \ldots + c_p \binom{\lambda+q+p}{q}, \tag{1}$$

the c's being arbitrary constants.

In the present paper we discuss the solution of the more general equations

$$\sum_{q=0}^{\infty} \frac{\alpha_q}{\lambda - r + q} = \gamma_r \quad (r = 0, 1, 2, \ldots) \tag{I}$$

in the unknowns $\alpha_0, \alpha_1, \alpha_2, \ldots$; as before, λ is a real constant, not an integer. We show that, provided that the γ_r satisfy the condition

$$\sum_{t=1}^{\infty} \frac{|\gamma_t| \log t}{t} < \infty, \tag{2}$$

the equations are always soluble in the case $0 < \lambda < 1$ and their solution is given by the formula

$$\alpha_q = -\frac{\sin \pi\lambda}{\pi\lambda} \frac{\lambda(\lambda+1)\ldots(\lambda+q)}{q!} \sum_{t=0}^{\infty} \frac{(-\lambda)(1-\lambda)\ldots(t-\lambda)}{t!} \frac{\gamma_t}{\lambda - t + q} \tag{II}$$

and is unique. When $\lambda < 0$, the equations are always soluble provided that (2) holds, and (II) is a solution whenever the series on the right are convergent, but the solution is not unique. The most general solution is obtained by adding to any particular solution the expression on the right of (1). If, more particularly, $\lambda < -1$, then the convergence of the series (II) is alone sufficient to ensure that (II) is a solution of (I). Lastly we show that when $\lambda > 1$ the equa-

† *Quart. J. of Math.* (Oxford), 10 (1939), 1–10, referred to as (E).

tions are soluble—under the condition (2)—if and only if the γ_t satisfy the $[\lambda]$ 'consistency conditions'

$$\sum_{r=0}^{\infty} \frac{(l+1-\lambda)...(l+r-\lambda)}{r!} \gamma_{l+r} = 0 \quad (l = 0, 1,..., [\lambda]-1)$$

and their unique solution is then given by (II). By writing

$$A_q = \frac{\pi}{\sin \pi\lambda} \frac{q!}{\Gamma(q+\lambda+1)} \alpha_q,$$

equations (I), (II) can be thrown into the reciprocal form, which is strikingly analogous to an inversion formula due to Titchmarsh,†

$$\left.\begin{aligned} \gamma_m &= \frac{\sin \pi\lambda}{\pi} \sum_{n=0}^{\infty} \frac{\Gamma(n+\lambda+1)}{n!} \frac{A_n}{\lambda+n-m} \\ A_m &= -\frac{\sin \pi\lambda}{\pi} \sum_{n=0}^{\infty} \frac{\Gamma(n-\lambda+1)}{n!} \frac{\gamma_n}{-\lambda+n-m} \end{aligned}\right\}. \qquad \text{(III)}$$

The paper in its present form owes much to the friendly criticisms and suggestions of Mr. T. W. Chaundy. Theorem A was originally proved by us under the stronger assumption $\sum_{t=1}^{\infty} |\gamma_t| t^{-\frac{1}{2}-\frac{1}{2}\lambda} < \infty$, while § 2 contained some rather cumbersome algebra. Mr. Chaundy showed us how to improve this algebra (the present § 2 is due to him) and this enabled us to sharpen the main theorems of the paper by relaxing the conditions on the γ's. Later, he supplied us with a great simplification of the analysis in § 3.

2. The following lemma may be proved without difficulty:

LEMMA 1. *The determinant*

$$\begin{bmatrix} a_0, a_1,..., a_N \\ b_0, b_1,..., b_N \end{bmatrix} = \left\| \frac{1}{a_r+b_s} \right\| \quad (r, s = 0, 1,..., N),$$

in which every $a_r+b_s \neq 0$, *is equal to the product*

$$\prod_{0 \leqslant r < s \leqslant N} (a_s-a_r) \prod_{0 \leqslant r < s \leqslant N} (b_s-b_r) \Big/ \prod_{r,s=0}^{N} (a_r+b_s).$$

Now consider the set of $(N+1)$ equations

$$\sum_{q=0}^{N} \frac{\alpha_q^{(N)}}{\lambda-r+q} = \gamma_r \quad (r = 0, 1,..., N) \qquad \text{(I)}_N$$

† E. C. Titchmarsh, *Proc. London Math. Soc.* (2) 26 (1927), 1–11.

in the unknowns $\alpha_0^{(N)}, \alpha_1^{(N)},..., \alpha_N^{(N)}$. Their solution is

$$\alpha_q^{(N)} = \sum_{t=0}^{N} \frac{\Delta_{tq}}{\Delta} \gamma_t \quad (q = 0, 1,..., N), \tag{3}$$

where Δ is the determinant

$$\begin{bmatrix} 0, 1,..., N \\ \lambda, \lambda-1,..., \lambda-N \end{bmatrix}$$

and Δ_{tq} is the cofactor of the term in the $(t+1)$th row and $(q+1)$th column. Thus

$$\frac{\Delta_{tq}}{\Delta} = (-1)^{t+q} \begin{bmatrix} 0, 1,..., q-1, q+1,..., N \\ \lambda, \lambda-1,..., \lambda-t+1, \lambda-t-1,..., \lambda-N \end{bmatrix} \Big/ \begin{bmatrix} 0, 1,..., N \\ \lambda, \lambda-1,..., \lambda-N \end{bmatrix}$$

$$= -\frac{\lambda(\lambda+1)...(\lambda+q)}{q!} \frac{(-\lambda)(1-\lambda)...(t-\lambda)}{t!} \frac{1}{\lambda+q-t} \times$$

$$\times \prod_{s=q+1}^{N} \left(1-\frac{\lambda}{s-q}\right) \prod_{r=t+1}^{N} \left(1+\frac{\lambda}{r-t}\right)$$

by an easy calculation. Thus

$$\alpha_q^{(N)} = \sum_{t=0}^{N} c_{tq}^{(N)} \gamma_t \quad (q = 0, 1,..., N), \tag{4}$$

where

$$c_{tq}^{(N)} = \frac{\Delta_{tq}}{\Delta} = -\frac{\lambda(\lambda+1)...(\lambda+q)}{q!} \frac{(-\lambda)(1-\lambda)...(t-\lambda)}{t!} \frac{1}{\lambda+q-t} \times$$

$$\times \prod_{s=1}^{N-q} \left(1-\frac{\lambda}{s}\right) \prod_{s=1}^{N-t} \left(1+\frac{\lambda}{s}\right).$$

Evidently $\lim_{N\to\infty} c_{tq}^{(N)} = c_{tq}$ exists for every t, q and

$$c_{tq} = -\frac{\sin \pi\lambda}{\pi\lambda} \frac{\lambda(\lambda+1)...(\lambda+q)}{q!} \frac{(-\lambda)...(t-\lambda)}{t!} \frac{1}{\lambda+q-t}. \tag{5}$$

3. On making $N \to \infty$, equations $(\mathrm{I})_N$ pass over formally into equations (I). We therefore define formally

$$\alpha_q^* = \sum_{t=0}^{\infty} c_{tq} \gamma_t \quad (q = 0, 1, 2,...) \tag{6}$$

and shall prove

THEOREM A. *If* $0 < \lambda < 1$ *and if*

$$\sum_{t=1}^{\infty} |\gamma_t| \frac{\log t}{t} < \infty, \tag{7}$$

then the series (5)—*that is, the series* (II) *of* § 1—*are convergent and the* α_q^* *are a solution of* (I). *Further,*

$$\sum_{q=1}^{\infty} \frac{|\alpha_q^*|}{q} < \infty. \tag{8}$$

Proof. To prove the convergence of the series (6), we observe that

$$c_{tq} = O_q\{(t+1)^{-1-\lambda}\}.$$

Hence
$$\sum_{t=0}^{\infty} |c_{tq}\gamma_t| = O_q\Big(\sum_{t=0}^{\infty} (t+1)^{-1-\lambda}|\gamma_t|\Big)$$

and the last series converges by (7).

To prove that the α_q^* satisfy (I) is equivalent to proving that

$$\sum_{q=0}^{\infty} \frac{\lambda...(\lambda+q)}{q!} \frac{1}{\lambda-r+q} \sum_{t=0}^{\infty} \frac{(1-\lambda)...(t-\lambda)}{t!} \frac{\gamma_t}{\lambda-t+q} = \frac{\pi}{\sin\pi\lambda}\gamma_r \qquad (r = 0, 1, 2,...) \tag{9}$$

First consider the formally inverted series

$$\sum_{t=0}^{\infty} \frac{(1-\lambda)...(t-\lambda)}{t!}\gamma_t \sum_{q=0}^{\infty} \frac{\lambda...(\lambda+q)}{q!} \frac{1}{(\lambda-r+q)(\lambda-t+q)} = \sum_{t=0}^{\infty} C_{rt}\gamma_t, \tag{10}$$

say; the inner sums are easily seen to be convergent.

For $t \geqslant 1$

$$\begin{aligned} C_{0t} &= (-1)^t \sum_{q=0}^{\infty} \frac{(\lambda-t)...(\lambda-t+q-1)}{q!} \frac{(\lambda-t+q+1)...(\lambda+q-1)}{t!} \\ &= \frac{(-1)^t}{t!}\big[D^{t-1}\{(1-x)^{t-\lambda}x^{\lambda-1}\}\big]_{x=1} \\ &= 0. \end{aligned}$$

Therefore

$$\sum_{q=0}^{\infty} \frac{\lambda...(\lambda+q)}{q!} \frac{t}{(\lambda+q)(\lambda-t+q)} = 0 \quad (t = 0, 1, 2,...) \tag{11}$$

and also

$$\sum_{q=0}^{\infty} \frac{\lambda...(\lambda+q)}{q!} \frac{r}{(\lambda+q)(\lambda-r+q)} = 0 \quad (r = 0, 1, 2,...).$$

Subtracting, we obtain

$$(t-r)\sum_{q=0}^{\infty} \frac{\lambda...(\lambda+q)}{q!} \frac{1}{(\lambda-r+q)(\lambda-t+q)} = 0 \quad (t, r = 0, 1, 2,...).$$

Thus
$$C_{rt} = 0 \quad (t \neq r;\quad t, r \geqslant 0).$$

To discuss the case $t = r \geqslant 0$, we write

$$k_r = \sum_{q=0}^{\infty} \frac{\lambda ... (\lambda+q)}{q!} \frac{1}{(\lambda-r+q)^2}$$

$$= \sum_{q=0}^{\infty} \frac{\lambda ... (\lambda+q-1)}{q!} \frac{q}{(\lambda-r+q-1)^2},$$

adding in a zero term at the beginning. Then

$$(r+1)k_r-(r+1-\lambda)k_{r+1}$$

$$= \sum_{q=0}^{\infty} \frac{\lambda ... (\lambda+q-1)}{q!} \frac{q(r+1)-(q+\lambda)(r+1-\lambda)}{(\lambda-r+q-1)^2}$$

$$= \lambda \sum_{q=0}^{\infty} \frac{\lambda ... (\lambda+q)}{q!} \frac{1}{(q+\lambda)(\lambda-r+q-1)}$$

$$= 0 \quad (r = 0, 1, 2, ...),$$

by (11). Therefore $$k_{r+1} = \frac{r+1}{r+1-\lambda} k_r,$$

which gives $$k_r = \frac{r!}{(r-\lambda)...(1-\lambda)} k_0.$$

Here

$$k_0 = \sum_{q=0}^{\infty} \frac{\lambda ... (\lambda+q-1)}{q!} \frac{1}{\lambda+q} = \int_0^1 (1-x)^{-\lambda} x^{\lambda-1}\, dx = \frac{\pi}{\sin \pi\lambda}.$$

Thus $$k_r = \frac{r!}{(1-\lambda)...(r-\lambda)} \frac{\pi}{\sin \pi\lambda}$$

and hence $$C_{rt} = \begin{cases} 0 & (t \neq r;\ t, r \geqslant 0), \\ \dfrac{\pi}{\sin \pi\lambda} & (t = r \geqslant 0). \end{cases} \tag{12}$$

The right-hand side of (10) thus reduces to

$$\frac{\pi}{\sin \pi\lambda} \gamma_r,$$

and equation (9) will therefore be established if we can justify the formal inversion of its left-hand side. To do this, we write

$$\sum_{t=0}^{\infty} \sum_{q=0}^{\infty} = \sum_{t=0}^{r} \sum_{q=0}^{\infty} + \sum_{t=r+1}^{\infty} \sum_{q=0}^{r-1} + \sum_{t=r+1}^{\infty} \sum_{q=r}^{t-1} + \sum_{t=r+1}^{\infty} \sum_{q=t}^{\infty}$$

$$= S_1+S_2+S_3+S_4$$

and show that each of these repeated sums can be inverted, i.e. is itself convergent and equal to the corresponding inverted sum.

The inversion of S_1 offers no difficulty, since the inner sum is convergent. S_2 can be inverted if

$$\sum_{t=r+1}^{\infty} \frac{(1-\lambda)...(t-\lambda)}{t!} \frac{1}{\lambda-t+q}\gamma_t$$

converges for every q; this is the case, since the dominating series $\sum_{t=1}^{\infty} |\gamma_t| t^{-1-\lambda}$ converges by (7).

In S_3 and S_4, inversion is possible if the corresponding double series are absolutely convergent; that is to say if, on replacing γ_t by $|\gamma_t|$, the resulting double series of positive terms S_3^* and S_4^* are convergent.

Convergence of S_3^*. Let

$$\phi(t) = \frac{(1-\lambda)...(t-\lambda)}{t!} \sum_{q=r}^{t-1} \frac{\lambda...(\lambda+q)}{q!} \frac{1}{(\lambda-r+q)(\lambda-t+q)}$$

denote the coefficient of $|\gamma_t|$ in S_3^*. Then S_3^* will converge if

$$\sum_{t=r+1}^{\infty} \phi(t)\, |\gamma_t| < \infty. \tag{13}$$

Now
$$\frac{(1-\lambda)...(t-\lambda)}{t!} < A(\lambda)t^{-\lambda},$$
while

$$\sum_{q=r}^{t-1} \frac{\lambda...(\lambda+q)}{q!} \frac{1}{(\lambda-r+q)(\lambda-t+q)} < A(r,\lambda) \sum_{q=r}^{t-1} q^{-1+\lambda} \frac{1}{t-\lambda-q}$$
$$< A(r,\lambda)t^{-1+\lambda}\log t$$

by an easy estimation. Thus, as $t \to \infty$,

$$\phi(t) < A(r,\lambda)t^{-1}\log t.$$

By (7), it follows that (13) holds and S_3^* is convergent.

Convergence of S_4^*. Let

$$\psi(t) = \frac{(1-\lambda)...(t-\lambda)}{t!} \sum_{q=t}^{\infty} \frac{\lambda...(\lambda+q)}{q!} \frac{1}{(\lambda-r+q)(\lambda-t+q)}$$

denote the coefficient of $|\gamma_t|$ in S_4^*. Then S_4^* will converge if

$$\sum_{t=0}^{\infty} \psi(t)\, |\gamma_t| < \infty. \tag{14}$$

Now for $t \geqslant 3$

$$\sum_{q=t}^{\infty} \frac{\lambda...(\lambda+q)}{q!} \frac{1}{\lambda-r+q} \frac{1}{\lambda-t+q} < A(r) \sum_{q=t}^{\infty} q^{-1+\lambda} \frac{1}{\lambda-t+q}$$

$$< A(r,\lambda) \sum_{q=t}^{\infty} q^{-1+\lambda} \frac{1}{q+1-t} = A(r,\lambda)\Big(\sum_{q=t}^{2t} + \sum_{q=2t+1}^{\infty}\Big)$$

$$< A(r,\lambda)(t^{-1+\lambda}\log t + t^{-1+\lambda}) < A(r,\lambda)t^{-1+\lambda}\log t.$$

Thus, as $t \to \infty$, $\qquad \psi(t) < A(r,\lambda)t^{-1}\log t;$

whence, by (7), $\qquad \sum_{t=0}^{\infty} \psi(t)\, |\gamma_t| < \infty$

and S_4^* is convergent.

This completes the proof that the α_q^* satisfy (I).

It remains to prove that

$$\sum_{q=1}^{\infty} \frac{|\alpha_q^*|}{q} < \infty.$$

From the trivial inequality

$$c_{tq} = O\left\{\left(\frac{q+1}{t+1}\right)^{\lambda} \frac{1}{1+|t-q|}\right\} \quad (t, q \geqslant 0)$$

we have

$$\sum_{q=0}^{\infty} \frac{|\alpha_q^*|}{q+1} \leqslant \sum_{q=0}^{\infty} \frac{1}{q+1} O\left\{\sum_{t=0}^{\infty} \left(\frac{q+1}{t+1}\right)^{\lambda} \frac{|\gamma_t|}{1+|t-q|}\right\}$$

$$= O\left\{\sum_{t=0}^{\infty} \frac{|\gamma_t|}{(t+1)^{\lambda}} \sum_{q=0}^{\infty} \frac{(q+1)^{\lambda-1}}{1+|t-q|}\right\}.$$

The inner sum

$$\sum_{q=0}^{\infty} = \sum_{q=0}^{t} + \sum_{q=t+1}^{\infty} = O(t^{\lambda-1}\log t) + O\left(\sum_{q=t+1}^{\infty} \frac{(q+1)^{\lambda-1}}{q-t}\right)$$

$$= O(t^{\lambda-1}\log t) \quad (t \geqslant 3).$$

Thus $\qquad \sum_{q=0}^{\infty} \frac{|\alpha_q^*|}{q+1} = O\left(|\gamma_0| + |\gamma_1| + |\gamma_2| + \sum_{t=3}^{\infty} \frac{\log t}{t} |\gamma_t|\right)$

$$< \infty,$$

and therefore $\qquad \sum_{q=1}^{\infty} \frac{|\alpha_q^*|}{q} < \infty.$

This completes the proof of Theorem A.

4. If $-1 < \lambda < 0$, then we may introduce the new equation

$$\frac{\alpha_0}{\lambda+1}+\frac{\alpha_1}{\lambda+2}+\ldots = \gamma_{-1},$$

where γ_{-1} is at our disposal. This, together with the equations (I), gives a new set

$$\sum_{q=0}^{\infty} \frac{\alpha_q}{\lambda-r+q} = \gamma_r \quad (r = -1, 0, 1, 2,\ldots),$$

which, provided that $\sum_{t=1}^{\infty} t^{-1}\log t|\gamma_t| < \infty$, has (by §§ 2, 3 with $\lambda+1$ for λ, γ_{t-1} for γ_t) the unique solution

$$\alpha_q = -\frac{\sin\pi(\lambda+1)}{\pi(\lambda+1)} \frac{(\lambda+1)\ldots(\lambda+1+q)}{q!} \times$$

$$\times \sum_{t=0}^{\infty} \frac{(-\lambda-1)(-\lambda)\ldots(t-\lambda-1)}{t!} \frac{\gamma_{t-1}}{\lambda+1-t+q}$$

$$= -\frac{\sin\pi\lambda}{\pi\lambda} \frac{\lambda(\lambda+1)\ldots(\lambda+q)}{q!} \sum_{r=0}^{\infty} \frac{(-\lambda)\ldots(r-\lambda)}{r!} \frac{(\lambda+q+1)\gamma_r}{(r+1)(\lambda-r+q)} -$$

$$-\frac{\sin\pi\lambda}{\pi\lambda} \frac{\lambda(\lambda+1)\ldots(\lambda+q)}{q!} \gamma_{-1}$$

$$= -\frac{\sin\pi\lambda}{\pi} \frac{(\lambda+1)\ldots(\lambda+q)}{q!} \sum_{t=0}^{\infty} \frac{(-\lambda)\ldots(t-1-\lambda)}{t!} \gamma_{t-1} -$$

$$-\frac{\sin\pi\lambda}{\pi\lambda} \frac{\lambda\ldots(\lambda+q)}{q!} \sum_{r=0}^{\infty} \frac{(-\lambda)\ldots(r-\lambda)}{r!} \frac{\gamma_r}{\lambda-r+q}$$

$$= \alpha_q^{(1)}+\alpha_q^{*}, \quad \text{say,} \tag{15}$$

provided that at least one of these series is convergent. Now the first term $\alpha_q^{(1)}$ is of the form $c\dfrac{(\lambda+1)\ldots(\lambda+q)}{q!}$, where c is independent of q; hence, by (E), we have

$$\sum_{q=0}^{\infty} \frac{\alpha_q^{(1)}}{\lambda-t+q} = 0 \quad (t = 0, 1, 2,\ldots).$$

It follows that, for $t = 0, 1, 2,\ldots$,

$$\sum_{q=0}^{\infty} \frac{\alpha_q^{*}}{\lambda-t+q} = \sum_{q=0}^{\infty} \frac{\alpha_q}{\lambda-t+q} - \sum_{q=0}^{\infty} \frac{\alpha_q^{(1)}}{\lambda-t+q} = \gamma.$$

That is to say, α_q^* is a solution of equations (I) with $-1 < \lambda < 0$, provided that

$$\text{(i)} \quad \sum_{t=1}^{\infty} t^{-1} \log t |\gamma_t| < \infty;$$

$$\text{(ii)} \quad \sum_{r=0}^{\infty} \frac{(-\lambda)...(r-\lambda)}{(r+1)!} \gamma_r \text{ converges.}$$

Next suppose $-2 < \lambda < -1$. We introduce two new equations and consider the set

$$\sum_{q=0}^{\infty} \frac{\alpha_q}{\lambda - r + q} = \gamma_r \quad (r = -2, -1, 0, ...),$$

where γ_{-2}, γ_{-1} are at our disposal. By §§ 2, 3 with $\lambda+2$ for λ and provided that $\sum_{t=1}^{\infty} t^{-1} \log t |\gamma_t| < \infty$, these equations have the unique solution

$$\alpha_q = -\frac{\sin \pi(\lambda+2)}{\pi(\lambda+2)} \frac{(\lambda+2)...(\lambda+2+q)}{q!} \times$$
$$\times \sum_{t=0}^{\infty} \frac{(-\lambda-2)...(t-\lambda-2)}{t!} \frac{\gamma_{t-2}}{\lambda+2-t+q},$$

which, by proceeding as before, can be written in the form

$$-\frac{\sin \pi\lambda}{\pi\lambda} \frac{\lambda...(\lambda+q)}{q!} \sum_{r=0}^{\infty} \frac{(-\lambda)...(r-\lambda)}{r!} \frac{\gamma_r}{\lambda-r+q} -$$
$$-\frac{\sin \pi\lambda}{\pi} \frac{(\lambda+1)...(\lambda+q)}{q!} \sum_{r=-1}^{\infty} \frac{(-\lambda)...(r-\lambda)}{(r+1)!} \gamma_r +$$
$$+\frac{\sin \pi\lambda}{\pi} \frac{(\lambda+2)...(\lambda+q+1)}{q!} \sum_{r=-2}^{\infty} \frac{(-1-\lambda)...(r-\lambda)}{(r+2)!} \gamma_r,$$

provided that the last two series converge; this will be so if $\sum_{r=0}^{\infty} \frac{(-\lambda)...(r-\lambda)}{(r+1)!} \gamma_r$ converges. Let us write this

$$\alpha_q = \alpha_q^* + \alpha_q^{(2)} + \alpha_q^{(3)}.$$

Since $\alpha_q^{(2)}$ and $\alpha_q^{(3)}$ are, by (E), both solutions of the equations

$$\sum_{q=0}^{\infty} \frac{\alpha_q}{\lambda-t+q} = 0 \quad (t = 0, 1, 2, ...), \tag{16}$$

it follows as before that α_q^* is a solution of (I) when $-2 < \lambda < -1$ provided that $\sum_{t=1}^{\infty} t^{-1}\log t|\gamma_t| < \infty$ and

$$\sum_{r=0}^{\infty} \frac{(-\lambda)...(r-\lambda)}{(r+1)!}\gamma_r$$

is convergent. The latter condition implies the former, since it gives $\frac{(-\lambda)...(r-\lambda)}{(r+1)!}\gamma_r = O(1)$, $\frac{|\gamma_r|\log r}{r} = O(r^{-|\lambda|}\log r)$. Thus in this case (I) possesses the solution (II) provided that $\sum_{r=0}^{\infty} \frac{(-\lambda)...(r-\lambda)}{(r+1)!}\gamma_r$ converges, or (which is equivalent) provided that the series on the right of (II) are convergent.

In general, we obtain

THEOREM B. *The equations* (I) *possess the solution* (II):

(i) *when* $\lambda < -1$ *provided that* $\sum_{t=0}^{\infty} \frac{(-\lambda)...(t-\lambda)}{(t+1)!}\gamma_t$ *converges or* (*which is equivalent*) *provided that the series in* (II) *are convergent*;

(ii) *when* $-1 < \lambda < 0$ *provided that* $\sum_{t=0}^{\infty} \frac{(-\lambda)...(t-\lambda)}{(t+1)!}\gamma_t$ *converges and also* $\sum_{t=1}^{\infty} t^{-1}\log t|\gamma_t| < \infty$.

In either case, their solution is not unique, and, if α_q^* *denotes the particular solution* (II), *their most general solution is*

$$\alpha_q = \alpha_q^* + c_0\binom{\lambda+q}{q} + c_1\binom{\lambda+q+1}{q} + ... + c_p\binom{\lambda+q+p}{q},$$

where $c_0,..., c_p$ *are arbitrary constants and* p *denotes the integer part of* $|\lambda|$.

The proof of the first part is a straightforward induction from the case $-k < \lambda < -k+1$ to the case $-k-1 < \lambda < -k$. That the most general solution is of the form stated then follows, by (E), from the fact that the difference of any two solutions of (I) is a solution of (16), and conversely.

5. If $1 < \lambda < 2$ and $\sum_{t=1}^{\infty} t^{-1}\log t|\gamma_t| < \infty$, equations (I) with the first equation omitted possess, by §§ 2, 3 with $\lambda-1$ for λ, the unique solution

$$\alpha_q = -\frac{\sin\pi(\lambda-1)}{\pi(\lambda-1)}\frac{(\lambda-1)...(\lambda+q-1)}{q!}\sum_{t=0}^{\infty}\frac{(1-\lambda)...(t+1-\lambda)}{t!}\frac{\gamma_{t+1}}{\lambda-t-1+q}. \tag{17}$$

The condition that this should also satisfy the first equation is

$$\gamma_0 = \sum_{q=0}^{\infty}\frac{1}{\lambda+q}\frac{-\sin\pi(\lambda-1)}{\pi(\lambda-1)}\frac{(\lambda-1)...(\lambda+q-1)}{q!}\times$$
$$\times\sum_{t=0}^{\infty}\frac{(1-\lambda)...(t+1-\lambda)}{t!}\frac{\gamma_{t+1}}{\lambda-t-1+q} \tag{18}$$
$$= -\frac{\sin\pi\lambda}{\pi\lambda}\sum_{t=0}^{\infty}\frac{(-\lambda)(1-\lambda)...(t+1-\lambda)}{(t+1)!}\gamma_{t+1}\times$$
$$\times\sum_{q=0}^{\infty}\frac{t+1}{(\lambda+q)(\lambda-t-1+q)}\frac{\lambda(\lambda+1)...(\lambda+q-1)}{q!} \tag{19}$$

since (as may be verified by a straightforward estimation) the condition $\sum_{t=1}^{\infty} t^{-1}\log t|\gamma_t| < \infty$ ensures the absolute convergence of the double series for $0 < \lambda-1 < 1$. The inner sum may be written formally as

$$\sum_{q=0}^{\infty}\frac{\lambda(\lambda+1)...(\lambda+q-1)}{q!}\frac{1}{\lambda-t-1+q} - \sum_{q=0}^{\infty}\frac{\lambda(\lambda+1)...(\lambda+q-1)}{q!}\frac{1}{\lambda+q}$$
$$= \frac{1}{\lambda-t-1}F(\lambda,\lambda-t-1;\lambda-t;1) - \frac{1}{\lambda}F(\lambda,\lambda;\lambda+1;1).$$

These series converge for $\Re(\lambda) < 1$ and the value of the above expression for these values of complex λ is

$$\frac{1}{\lambda-t-1}\frac{\Gamma(\lambda-t)\Gamma(1-\lambda)}{\Gamma(-t)\Gamma(1)} - \frac{1}{\lambda}\frac{\Gamma(\lambda+1)\Gamma(1-\lambda)}{\Gamma(1)\Gamma(1)} = -\frac{\pi}{\sin\pi\lambda}.$$

By analytic continuation, it follows that the inner sum on the right of (19) still has this value for $1 < \lambda < 2$; (19) may therefore be written

$$\gamma_0 + \sum_{r=1}^{\infty}\frac{(1-\lambda)...(r-\lambda)}{r!}\gamma_r = 0. \tag{20}$$

If this 'consistency condition' is satisfied and if $\sum_{t=1}^{\infty} t^{-1}\log t|\gamma_t| < \infty$, then equations (I) have the unique solution (17). We ask whether their solution is still given also by (II). This will be so if, for $q = 0, 1, 2,\dots$,

$$-\frac{\sin\pi\lambda}{\pi\lambda}\frac{\lambda(\lambda+1)\dots(\lambda+q)}{q!}\sum_{t=0}^{\infty}\frac{(-\lambda)\dots(t-\lambda)}{t!}\frac{\gamma_t}{\lambda-t+q}$$

$$= -\frac{\sin\pi(\lambda-1)}{\pi(\lambda-1)}\frac{(\lambda-1)\lambda\dots(\lambda+q-1)}{q!}\sum_{t=0}^{\infty}\frac{(1-\lambda)\dots(t+1-\lambda)}{t!}\frac{\gamma_{t+1}}{\lambda-t-1+q},$$

i.e. if

$$\sum_{t=0}^{\infty}\frac{(1-\lambda)\dots(t-\lambda)}{t!}\frac{\gamma_t}{\lambda-t+q} = \sum_{r=1}^{\infty}\frac{(1-\lambda)\dots(r-\lambda)}{r!}\left(\frac{\gamma_r}{\lambda-r+q}-\frac{\gamma_r}{\lambda+q}\right),$$

i.e. if
$$\frac{1}{\lambda+q}\sum_{r=0}^{\infty}\frac{(1-\lambda)\dots(r-\lambda)}{r!}\gamma_r = 0,$$

which is so by (20). Hence: *If* $1 < \lambda < 2$ *and* $\sum_{t=1}^{\infty} t^{-1}\log t|\gamma_t| < \infty$, (I) *has no solution unless* (20) *is satisfied, when it has the unique solution* (II).

More generally we have

THEOREM C. *Suppose that* $1 \leqslant k < \lambda < k+1$ (*k an integer*), *and that*
$$\sum_{t=1}^{\infty} t^{-1}\log t|\gamma_t| < \infty.$$
Then equations (I) *are soluble if and only if the γ_t satisfy the k consistency conditions*

$$\sum_{r=0}^{\infty}\frac{(l+1-\lambda)\dots(l+r-\lambda)}{r!}\gamma_{l+r} = 0 \quad (l = 0, 1,\dots, k-1). \tag{21}$$

Their unique solution is then given by (II) *and satisfies the condition*

$$\sum_{q=1}^{\infty}\frac{|\alpha_q|}{q} < \infty. \tag{22}$$

Proof. That the solution, if it exists, is unique follows from (E). To prove the rest of the theorem we use an induction argument:

1. The theorem is already established in the case $1 < \lambda < 2$.

2. Suppose that $k > 1$ and that the theorem is already established in the cases $s < \lambda < s+1$; $s = 1, 2,\dots, k-1$. We shall show that it follows in the case $k < \lambda < k+1$.

2.1. First suppose that (21) holds. Then in particular

$$\sum_{r=0}^{\infty} \frac{(l+2-\lambda)...(l+r+1-\lambda)}{r!} \gamma_{l+r+1} = 0 \quad (l = 0, 1, ..., k-2)$$

and hence, since $k-1 < \lambda-1 < k$, it follows from our induction hypothesis that the equations

$$\sum_{q=0}^{\infty} \frac{\alpha_q}{\lambda-r+q} = \gamma_r \quad (r = 1, 2, ...)$$

possess the solution

$$\alpha_q = -\frac{\sin\pi(\lambda-1)}{\pi(\lambda-1)} \frac{(\lambda-1)\lambda...(\lambda+q-1)}{q!} \times$$
$$\times \sum_{r=0}^{\infty} \frac{(1-\lambda)...(r+1-\lambda)}{r!} \frac{\gamma_{r+1}}{\lambda-r-1+q}. \quad (23)$$

They will therefore possess the solution (II) if

$$-\frac{\sin\pi\lambda}{\pi\lambda} \frac{\lambda...(\lambda+q)}{q!} \sum_{t=0}^{\infty} \frac{(-\lambda)...(t-\lambda)}{t!} \frac{\gamma_t}{\lambda-t+q}$$
$$= -\frac{\sin\pi(\lambda-1)}{\pi(\lambda-1)} \frac{(\lambda-1)...(\lambda+q-1)}{q!} \sum_{t=0}^{\infty} \frac{(1-\lambda)...(t+1-\lambda)}{t!} \frac{\gamma_{t+1}}{\lambda-t-1+q},$$

which has already been seen to reduce to

$$\frac{1}{\lambda+q} \sum_{r=0}^{\infty} \frac{(1-\lambda)...(r-\lambda)}{r!} \gamma_r = 0,$$

and this is true by (21) with $l = 0$. The condition that the remaining equation ($r = 0$) of (I) should also be satisfied by (23), and therefore by (II), is

$$\gamma_0 = \sum_{q=0}^{\infty} \frac{1}{\lambda+q} \frac{-\sin\pi(\lambda-1)}{\pi(\lambda-1)} \frac{(\lambda-1)\lambda...(\lambda+q-1)}{q!} \times$$
$$\times \sum_{r=0}^{\infty} \frac{(1-\lambda)...(r+1-\lambda)}{r!} \frac{\gamma_{r+1}}{\lambda-r-1+q}. \quad (24)$$

Now

$$-\frac{\sin\pi(\lambda-1)}{\pi(\lambda-1)} \frac{(\lambda-1)\lambda...(\lambda+q-1)}{q!} \sum_{r=0}^{\infty} \frac{(1-\lambda)...(r+1-\lambda)}{r!} \frac{\gamma_{r+1}}{\lambda-r-1+q}$$
$$= -\frac{\sin\pi(\lambda-k)}{\pi(\lambda-k)} \frac{(\lambda-k)...(\lambda+q-k)}{q!} \sum_{t=0}^{\infty} \frac{(k-\lambda)...(t+k-\lambda)}{t!} \frac{\gamma_{t+k}}{\lambda-t-k+q},$$

since each of these expressions, equated to α_q, gives the unique solution of equations (I) with the first k equations removed. Hence we may write (24) in the form

$$\gamma_0 = \sum_{q=0}^{\infty} \frac{1}{\lambda+q} \frac{-\sin\pi(\lambda-k)}{\pi(\lambda-k)} \frac{(\lambda-k)...(\lambda+q-k)}{q!} \times$$

$$\times \sum_{t=0}^{\infty} \frac{(k-\lambda)...(t+k-\lambda)}{t!} \frac{\gamma_{t+k}}{\lambda-t-k+q}$$

$$= \sum_{t=0}^{\infty} \gamma_{t+k} \frac{(k-\lambda)...(t+k-\lambda)}{t!} \frac{-\sin\pi(\lambda-k)}{\pi(\lambda-k)} \times$$

$$\times \sum_{q=0}^{\infty} \frac{1}{\lambda+q} \frac{(\lambda-k)...(\lambda+q-k)}{q!} \frac{1}{\lambda-k-t+q}, \tag{25}$$

since the double series may be verified to be absolutely convergent (for $k < \lambda < k+1$) in virtue of the condition $\sum_{t=1}^{\infty} t^{-1}\log t|\gamma_t| < \infty$,

$$= (-1)^{k-1}\frac{\sin\pi\lambda}{\pi} \sum_{t=0}^{\infty} \gamma_{t+k} \frac{(k-\lambda)...(t+k-\lambda)}{t!} S,$$

where

$$S = \sum_{q=0}^{\infty} \frac{(\lambda-k+1)...(\lambda-k+q)}{q!} \left(\frac{1}{\lambda-k-t+q} - \frac{1}{\lambda+q}\right) \frac{1}{t+q}$$

$$= \frac{\pi}{\sin\pi\lambda} \frac{(1-\lambda)...(k-1-\lambda)}{(k-1)!} \frac{1}{t+k}$$

by an argument entirely similar to that of p. 94. Thus (24) may be written

$$\gamma_0 = (-1)^k \frac{(1-\lambda)...(k-1-\lambda)}{(k-1)!} \sum_{t=0}^{\infty} \frac{\gamma_{t+k}}{t+k} \frac{(k-\lambda)...(t+k-\lambda)}{t!}$$

$$= \frac{(-1)^k}{(k-1)!} \sum_{r=k}^{\infty} \frac{(1-\lambda)...(r-\lambda)}{(r-k)!} \frac{\gamma_r}{r}.$$

But this is precisely the equation obtained on eliminating $\gamma_1,..., \gamma_{k-1}$ from equations (21). Hence the first of the equations (I) is also satisfied by (II), i.e. the set (I) possesses the solution (II).

2.2. Conversely, suppose that equations (I) do possess the solution (II). Then so will equations (I) with the first equation omitted. By our induction hypothesis it follows that the consistency conditions

$$\sum_{r=0}^{\infty} \frac{(l+2-\lambda)...(l+r+1-\lambda)}{r!} \gamma_{l+r+1} = 0 \quad (l = 0, 1,..., k-2)$$

hold, i.e. we have

$$\sum_{r=0}^{\infty} \frac{(l+1-\lambda)...(l+r-\lambda)}{r!} \gamma_{l+r} = 0 \quad (l = 1, 2,..., k-1). \tag{26}$$

Further, the set (I) with the first equation omitted has for its unique solution

$$\alpha_q = -\frac{\sin \pi(\lambda-1)}{\pi(\lambda-1)} \frac{(\lambda-1)...(\lambda+q-1)}{q!} \sum_{r=0}^{\infty} \frac{(1-\lambda)...(r+1-\lambda)}{r!} \frac{\gamma_{r+1}}{\lambda-r-1+q}.$$

This is therefore also the only solution which the full set (I) can have, i.e. it is identical with (II). Equating these two expressions for α_q, we obtain, as before,

$$\frac{1}{\lambda+q} \sum_{r=0}^{\infty} \frac{(1-\lambda)...(r-\lambda)}{r!} \gamma_r = 0,$$

and thus (26) also holds when $l = 0$, i.e. (21) is satisfied.

The theorem is therefore proved by induction except for the property (22). This, however, follows at once on applying theorem A to the set (I) with the first k equations removed.

AN ALGEBRAIC PROPERTY OF LAPLACE'S DIFFERENTIAL EQUATION*

By OLGA TAUSSKY (*London*)

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It is known that each of a pair of functions satisfying the Cauchy-Riemann equations satisfies the Laplace equation; similarly for each of a set of functions satisfying the Dirac equations. The reason for this is given by the following relations.

Let u_1, u_2 be two functions of x_1, x_2.†

Put
$$l_1 \equiv \frac{\partial u_1}{\partial x_1} - \frac{\partial u_2}{\partial x_2}, \qquad l_2 \equiv \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}.$$

Then
$$\frac{\partial}{\partial x_1} l_1 + \frac{\partial}{\partial x_2} l_2 \equiv \frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_1}{\partial x_2^2},$$
$$-\frac{\partial}{\partial x_2} l_1 + \frac{\partial}{\partial x_1} l_2 \equiv \frac{\partial^2 u_2}{\partial x_1^2} + \frac{\partial^2 u_2}{\partial x_2^2}.$$

Similarly, let u_1, u_2, u_3, u_4 be four functions of x_1, x_2, x_3, x_4. Put
$$l_1 \equiv \frac{\partial u_1}{\partial x_1} - \frac{\partial u_2}{\partial x_2} - \frac{\partial u_3}{\partial x_3} - \frac{\partial u_4}{\partial x_4},$$
$$l_2 \equiv \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \pm \frac{\partial u_3}{\partial x_4} \mp \frac{\partial u_4}{\partial x_3},$$
$$l_3 \equiv \frac{\partial u_1}{\partial x_3} \mp \frac{\partial u_2}{\partial x_4} + \frac{\partial u_3}{\partial x_1} \pm \frac{\partial u_4}{\partial x_2},$$
$$l_4 \equiv \frac{\partial u_1}{\partial x_4} \pm \frac{\partial u_2}{\partial x_3} \mp \frac{\partial u_3}{\partial x_2} + \frac{\partial u_4}{\partial x_1}.$$

Then‡
$$\frac{\partial}{\partial x_1} l_1 + \frac{\partial}{\partial x_2} l_2 + \frac{\partial}{\partial x_3} l_3 + \frac{\partial}{\partial x_4} l_4 \equiv \sum_{r=1}^{4} \frac{\partial^2 u_1}{\partial x_r^2},$$
$$-\frac{\partial}{\partial x_2} l_1 + \frac{\partial}{\partial x_1} l_2 \mp \frac{\partial}{\partial x_4} l_3 \pm \frac{\partial}{\partial x_3} l_4 \equiv \sum_{r=1}^{4} \frac{\partial^2 u_2}{\partial x_r^2},$$
$$-\frac{\partial}{\partial x_3} l_1 \pm \frac{\partial}{\partial x_4} l_2 + \frac{\partial}{\partial x_1} l_3 \mp \frac{\partial}{\partial x_2} l_4 \equiv \sum_{r=1}^{4} \frac{\partial^2 u_3}{\partial x_r^2},$$
$$-\frac{\partial}{\partial x_4} l_1 \mp \frac{\partial}{\partial x_3} l_2 \pm \frac{\partial}{\partial x_2} l_3 + \frac{\partial}{\partial x_1} l_4 \equiv \sum_{r=1}^{4} \frac{\partial^2 u_4}{\partial x_r^2}.$$

* See reference (**1**) at end of article. † (**2**). ‡ (**3**).

Similar relations hold for eight functions of eight variables. The linear forms*

$$l_j \equiv c^{hk}{}_j \frac{\partial u_k}{\partial x_h} \quad (h,\, k,\, j = 1,\ldots,\, 8),$$

have the same coefficient matrix $(c^{hk}{}_j)$ as the so-called Hurwitz-Brioschi relation.†

No other relations of this kind, however, exist except perhaps for the case of multiples of 16 (see Theorem III). In order to establish this we first show that the existence of such relations is intimately connected with the existence of hypercomplex systems without divisors of zero (Theorem I). Then we appeal to the theory of hypercomplex systems to obtain Theorem III. Theorem I can be considered as a special case of a theorem on the existence of certain isomorphisms of modules with real coefficients.

Let Δ denote the Laplacian operator

$$\Delta = a^{ij}\partial_i \partial_j \quad \left(\partial_i = \frac{\partial}{\partial x_i}, \quad \partial_j = \frac{\partial}{\partial x_j}\right),$$

where a^{ij} is a real and positive (or negative) definite tensor in n dimensions. Then the following theorem holds:

Theorem I. *Let*

$$\left.\begin{aligned} l_j &\equiv c^{hk}{}_j \partial_h u_k \quad (1) \\ \Delta u_k &\equiv b_k{}^{ij} \partial_i l_j \quad (2) \end{aligned}\right\} \quad (i,\, j,\, k,\, h = 1,\ldots,\, n),$$

where all the $c^{hk}{}_j$ and $b_k{}^{ij}$ are real constants. Then n must be such that there exists a hypercomplex system, not necessarily associative, with n base elements, with respect to the real numbers, without divisors of zero.‡

Theorem I is a consequence of the following:

Lemma. *Let*

$$\left.\begin{aligned} l_j &\equiv c^{hk}{}_j \partial_h u_k \quad (1\,\mathrm{a}) \\ \Delta u_k &\equiv b_k{}^{ij} \partial_i l_j \quad (2\,\mathrm{a}) \end{aligned}\right\} \quad \begin{pmatrix} h,\, i = 1,\ldots,\, n, \\ j = 1,\ldots,\, m, \\ k = 1,\ldots,\, l \end{pmatrix},$$

where all the $c^{hk}{}_j$ and $b_k{}^{ij}$ are real constants. Then three real modules M_n, M_l, M_m of n, l, m dimensions exist such that M_n is isomorphic with a module of isomorphisms of M_l on to a submodule of M_m, and that M_l is isomorphic with a module of isomorphisms of M_n on to a submodule of M_m.

This lemma can be proved in the following way. We first observe that in the relations (1), (2), (1 a), (2 a), we may replace the operators $\partial_i = \partial/\partial x_i$ by constant factors and consider these relations as

* (4). † (5). ‡ (6).

algebraic identities in u and ∂. This can be seen, for instance, if we consider the special functions

$$u_i = A_i e^{\partial_1 x_1 + \dots + \partial_n x_n},$$

where A_i, ∂_i are constants. We then obtain

$$\frac{\partial u_i}{\partial x_k} = u_i \partial_k, \qquad \frac{\partial^2 u_i}{\partial x_k \partial x_j} = u_i \partial_k \partial_j, \qquad \dots .$$

Let M_n, M_l, M_m be three real modules of dimensions n, l, m with base elements x^h, y^k, z^j respectively. Let us define the following multiplication:

$$x^h \times y^k = c^{hk}{}_j z^j, \tag{3}$$

$$p_h x^h \times q_k y^k = p_h q_k x^h y^k = p_h q_k c^{hk}{}_j z^j \text{ by (3)}, \tag{4}$$

p_h, q_k being any real numbers. By means of this multiplication every element of M_n defines a homomorphism of M_l on to a submodule of M_m. Hence M_n is homomorphic with a module of homomorphisms of M_l on to a submodule of M_m. Similarly, M_l is homomorphic with a module of homomorphisms of M_n on to a submodule of M_m.

What has to be proved now is that, if (1 a), (2 a) hold, these homomorphisms are isomorphisms, i.e. that $p_h q_k c^{hk}{}_j$ are not all zero, unless all the coordinates $p_h = 0$ or $q_k = 0$. Suppose this is not true. Then real numbers p_h, not all zero, and q_k, not all zero, exist, such that

$$p_h q_k c^{hk}{}_j = 0. \tag{5}$$

Substitute p_h, q_k in (1 a), (2 a) instead of ∂_h, u_k; then

$$l_j(p,q) = c^{hk}{}_j p_h q_k, \tag{1 b}$$

$$a^{hi} p_i p_h q_k = b_k{}^{ij} p_i l_j(p,q). \tag{2 b}$$

By (5), $l_j(p,q) = 0$, hence $a^{hi} p_i p_h q_k = 0$. Thus either all the coordinates $q_k = 0$, or $a^{hi} p_i p_h = 0$, which implies $p_h = 0$.

Theorem I follows from this lemma if we put $l = n = m$ and $x^h = y^h = z^h$, i.e. $M_l = M_n = M_m = M$. The module M together with the multiplication (3), (4) is a hypercomplex system. The fact that the homomorphisms mentioned in the lemma are isomorphisms means in this special case that M has no divisors of zero.

Another consequence of the lemma is

THEOREM II. *Let*

$$\left.\begin{aligned} l_j &\equiv c^{hk}{}_j \partial_h u_k \\ \Delta u_k &\equiv b_k{}^{ij} \partial_i l_j \end{aligned}\right\} \quad \begin{pmatrix} h, i = 1,\dots, n, \\ k, j = 1,\dots, m \end{pmatrix}, \qquad \begin{matrix}(1\,\text{c})\\(2\,\text{c})\end{matrix}$$

where all the $c^{hk}{}_j$ and $b_k{}^{ij}$ are real constants. Then the m-dimensional real vector group possesses an n-dimensional module of automorphisms.

This can be proved by putting $l = m$ and $M_l = M_m$ in the lemma. According to the lemma, M_n is isomorphic with a module of homomorphisms of M_m on to itself which are necessarily isomorphisms. Hence they are automorphisms.

The only known hypercomplex systems over the real numbers without divisors of zero are the real or complex numbers, the quaternions and the Cayley numbers.* The existence of other such systems is so far undecided, but it is known that the number of base elements of such a system is necessarily a multiple of 16. This follows from the following theorem which was proved independently by C. Ehresmann and E. Stiefel:† *The $(n-1)$-dimensional real projective space has $n-1$ linearly independent continuous vector functions only if $n = 1, 2, 4, 8, 16m$, where the possible values of m are not yet known.* An n-dimensional manifold with n linearly independent continuous vector functions is called parallelizable.‡ It can easily be seen that the $(n-1)$-dimensional projective space is necessarily parallelizable, if there exists a hypercomplex system, with n base elements, with respect to the real numbers, without divisors of zero.‖ Hence the number of the latter must be 1, 2, 4, 8, or $16m$. From this follows

THEOREM III. *Let*

$$\left.\begin{aligned} l_j &\equiv c^{hk}{}_j \partial_h u_k \\ \Delta u_k &\equiv b_k{}^{ij} \partial_i l_j \end{aligned}\right\} \quad (i, h, k, j = 1, \dots, n), \qquad \begin{aligned} &(1) \\ &(2) \end{aligned}$$

where all the $c^{hk}{}_j$ and $b_k{}^{ij}$ are real constants. Then n must be 1, 2, 4, 8, *or* $16m$, *where the possible values of m are not yet known.*

I am indebted to Professor D. van Dantzig for showing me how the methods of my own theorem could be used to establish a more general result.

REFERENCES

1. See my note in *Bull. American Math. Soc.* 44 (1938), 487. The problem discussed here has also also been considered by D. Iwanenko and K. Nikolsky, *Zeits. f. Physik*, 63 (1930), 129.
2. All functions u which will be considered are assumed to be defined in some given domain of the x-space and to have there everywhere continuous second derivatives.
3. Cf. R. Fueter, *C.R. Oslo* (1936), i. 76.
4. Here and elsewhere we are using the summation convention.
5. A. Hurwitz, *Math. Werke*, ii. 565–71, especially 570.

* (7). † (8). ‡ (9). ‖ (10).

6. It can be easily seen that it is not necessary to assume $c^{hk}{}_j$ to be real.

A not necessarily associative hypercomplex system with n base elements $e_1, \ldots, e_n$ with respect to the real numbers is the set of $x_1 e_1 + \ldots + x_n e_n$, where the x_i are real numbers. The sum of two elements $x_1 e_1 + \ldots + x_n e_n$ and $y_1 e_1 + \ldots + y_n e_n$ is defined as $(x_1+y_1)e_1 + \ldots + (x_n+y_n)e_n$, the product as $L_1(x,y)e_1 + \ldots + L_n(x,y)e_n$ where the $L_i(x,y) = C_{ijk} x_j y_k$ are any fixed bilinear forms in x, y with real coefficients. The product $r(x_1 e_1 + \ldots + x_n e_n)$ where r is a real number is $rx_1 e_1 + \ldots + rx_n e_n$. Two elements both of which have at least one coordinate different from zero, whose product has all its coordinates zero, are called zero divisors.

7. The first three results constitute Frobenius's theorem. That the Cayley numbers have no zero divisors was proved first by L. E. Dickson, *Trans. American Math. Soc.* 13 (1912), 72, or *Linear Algebras* (Cambridge 1916), 15.

8. C. Ehresmann, *J. de math.* IX, 16 (1937), 69–100; E. Stiefel, *Verh. d. Schweizer. Naturforsch. Ges.* (1935), 277–8.

9. See H. Hopf, *Enseign. Math.* 35 (1936), 345.

10. Professor Hopf indicated this relation between parallelizable projective spaces and hypercomplex systems without divisors of zero to me. Recently he informed me that it has now been proved that $n = 2^m$ are the only possible dimensions of such hypercomplex systems. Topological proofs of this last result are due to H. Hopf and to Stiefel; an algebraico-geometrical proof has been obtained by F. Behrend (to appear in *Compositio Math.*).

SUMMATION FORMULAE AND SELF-RECIPROCAL FUNCTIONS (II)

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1. Introduction

IN a previous paper* of the same title, referred to as (I), I discussed summation formulae for a certain class of sums of the form

$$\sum_{n=1}^{\infty} a_n f(n), \tag{1.1}$$

and I showed how these summation formulae could be expressed in the form of a Parseval theorem for functions of $L^2(0,\infty)$.

It has also been shown that

$$\sum_{n\leqslant x}' r_p(n) - \frac{\pi^{\frac{1}{2}p} x^{\frac{1}{2}p}}{\Gamma(1+\frac{1}{2}p)} = \sum_{n=1}^{\infty} r_p(n)\left(\frac{x}{n}\right)^{\frac{1}{2}p} J_{\frac{1}{2}p}(2\pi n^{\frac{1}{2}} x^{\frac{1}{2}}), \tag{1.2}$$

where $r_p(n)$ is the number of ways of expressing n as the sum of the squares of p integers, and the series on the right is summable $(R, n, \frac{1}{2}p-\frac{3}{2}+\epsilon)$ $(\epsilon > 0)$, but not summable $(R, n, \frac{1}{2}p-\frac{3}{2})$.† Similar results have also been proved for $\sigma_k(n)$, the sum of the kth powers of the divisors of n, and for $d_k(n)$, the number of ways of expressing n as the product of k positive integers.‡ These results suggest that there should be a class of summation formulae in which the series (1.1) does not converge, but is summable by the appropriate Riesz means.§ In this paper I use an extension of the method of (I) to establish such summation formulae for a large class of sequences $\{a_n\}$.

2. Formalities

The method can be illustrated by an example. Wilton‖ has shown that, if k is an integer such that $p \leqslant 4k+6$ and

$$P_k(x) = \frac{1}{\Gamma(k+1)} \sum_{1\leqslant n\leqslant x} r_p(n)(x-n)^k - \frac{\pi^{\frac{1}{2}p} x^{\frac{1}{2}p+k}}{\Gamma(1+\frac{1}{2}p+k)},$$

then $P_k(x)x^{-\frac{1}{2}p-\frac{1}{2}-k}$ is self-reciprocal with respect to the kernel

* A. P. Guinand, *Quart. J. of Math.* (Oxford), 9 (1938), 53–67.

† A. Oppenheim, *Proc. London Math. Soc.* (2), 26 (1927), 295–350.

‡ A. Oppenheim, loc. cit., and H. Cramer, *Arkiv för Math. och Fys.* 16 (1922), 21.

§ W. L. Ferrar, *Compositio Math.* 1 (1935), 344–60, has shown that such formulae do exist.

‖ J. R. Wilton, *Proc. London Math. Soc.* (2), 29 (1927), 168–88.

$\pi J_{\frac{1}{2}p+k}(2\pi x^{\frac{1}{2}})$. Further, it can be shown that, if $f(x)$ and $g(x)$ are a pair of transforms with respect to the kernel $\pi J_{\frac{1}{2}p}(2\pi x^{\frac{1}{2}})$, then

$$x^{\frac{1}{4}p+\frac{1}{4}+k}\left(\frac{d}{dx}\right)^{k+1}\{x^{-\frac{1}{4}p+\frac{1}{4}}f(x)\}, \qquad x^{\frac{1}{4}p+\frac{1}{4}+k}\left(\frac{d}{dx}\right)^{k+1}\{x^{-\frac{1}{4}p+\frac{1}{4}}g(x)\}$$

are formally a pair of transforms with respect to the kernel

$$\pi J_{\frac{1}{2}p+k}(2\pi x^{\frac{1}{2}}).$$

Hence we have formally, by the Parseval theorem for such transforms,

$$\int_0^\infty P_k(x)\left(\frac{d}{dx}\right)^{k+1}\{x^{-\frac{1}{4}p+\frac{1}{4}}f(x)\}\,dx = \int_0^\infty P_k(x)\left(\frac{d}{dx}\right)^{k+1}\{x^{-\frac{1}{4}p+\frac{1}{4}}g(x)\}\,dx. \quad (2.1)$$

The left-hand side is

$$\lim_{N\to\infty}\int_0^N P_k(x)\left(\frac{d}{dx}\right)^{k+1}\{x^{-\frac{1}{4}p+\frac{1}{4}}f(x)\}\,dx.$$

Integrating by parts $k+1$ times this becomes

$$\lim_{N\to\infty}(-)^k\left\{\left[\sum_{r=0}^{k}(-)^r P_r(x)\left(\frac{d}{dx}\right)^r\{x^{-\frac{1}{4}p+\frac{1}{4}}f(x)\}\right]_0^N - \right.$$

$$\left. -\int_0^N x^{-\frac{1}{4}p+\frac{1}{4}}f(x)\,d\left[\sum_{1\leqslant n\leqslant x} r_p(n)-\frac{\pi^{\frac{1}{2}p}x^{\frac{1}{2}p}}{\Gamma(1+\frac{1}{2}p)}\right]\right\}. \quad (2.2)$$

Now a function $S(N)$ is said to be limitable (R, n, κ) to S as N tends to infinity if

$$\lim_{N\to\infty}\kappa N^{-\kappa}\int_0^N (N-t)^{\kappa-1}S(t)\,dt = S.$$

Further, if $\lim\limits_{N\to\infty} S(N)$ exists, it is also equal to S.

If $f(x)$ satisfies suitable conditions, we can show that the integrated terms in (2.2) are limitable (R, n, κ) to zero for sufficiently large values of κ. The integral in (2.2) gives, apart from a factor $(-)^{k+1}$,

$$\lim_{N\to\infty}\left\{\kappa N^{-\kappa}\int_0^N (N-t)^{\kappa-1}\,dt\int_0^t x^{-\frac{1}{4}p+\frac{1}{4}}f(x)\,d\left[\sum_{1\leqslant n\leqslant x} r_p(n)-\frac{\pi^{\frac{1}{2}p}x^{\frac{1}{2}p}}{\Gamma(1+\frac{1}{2}p)}\right]\right\}$$

$$=\lim_{N\to\infty}\left\{N^{-\kappa}\int_0^N x^{-\frac{1}{4}p+\frac{1}{4}}f(x)(N-x)^{\kappa}\,d\left[\sum_{1\leqslant n\leqslant x} r_p(n)-\frac{\pi^{\frac{1}{2}p}x^{\frac{1}{2}p}}{\Gamma(1+\frac{1}{2}p)}\right]\right\}$$

$$=\lim_{N\to\infty}\left\{\sum_{n=1}^{N} r_p(n)n^{-\frac{1}{4}p+\frac{1}{4}}f(n)\left(1-\frac{n}{N}\right)^{\kappa}-\frac{\pi^{\frac{1}{2}p}}{\Gamma(\frac{1}{2}p)}\int_0^N x^{\frac{1}{4}p-\frac{1}{4}}f(x)\left(1-\frac{x}{N}\right)^{\kappa}dx\right\}.$$

Treating the right-hand side of (2.1) in the same way we have

$$\lim_{N\to\infty}\left\{\sum_{n=1}^{N} r_p(n)n^{-\frac{1}{4}p+\frac{1}{2}}f(n)\left(1-\frac{n}{N}\right)^{\kappa}-\frac{\pi^{\frac{1}{2}p}}{\Gamma(\frac{1}{2}p)}\int_0^N x^{\frac{1}{4}p-\frac{1}{2}}f(x)\left(1-\frac{x}{N}\right)^{\kappa}dx\right\}$$

$$=\lim_{N\to\infty}\left\{\sum_{n=1}^{N} r_p(n)n^{-\frac{1}{4}p+\frac{1}{2}}g(n)\left(1-\frac{n}{N}\right)^{\kappa}-\frac{\pi^{\frac{1}{2}p}}{\Gamma(\frac{1}{2}p)}\int_0^N x^{\frac{1}{4}p-\frac{1}{2}}g(x)\left(1-\frac{x}{N}\right)^{\kappa}dx\right\}.$$

This is a summation formula of the required form.

3. The self-reciprocal function

As in (I) let us suppose that there exists a sequence of real numbers $\{a_n\}$ and a function $R_0(x)$ which is expressible as the sum of a finite number of terms of the form $ax^m(\log x)^n$, where n is a non-negative integer. Let the greatest and least values of m be ρ and θ respectively, and put

$$R_k(x)=\frac{1}{\Gamma(k)}\int_0^{\infty}(x-t)^{k-1}R_0(t)\,dt. \tag{3.1}$$

Hence, for positive integral values of k, $R_k(x)$ will be the sum of a finite number of terms of the form $bx^r(\log x)^n$, where the greatest and least values of r are $\rho+k$ and $\theta+k$ respectively, and

$$R_k(x)=\begin{cases}O(x^{\rho+k+\epsilon}) & (x\to\infty), & (3.2)\\ O(x^{\theta+k-\epsilon}) & (x\to 0), & (3.21)\end{cases}$$

for any $\epsilon>0$. Let us put

$$\Delta_k(x)=\frac{1}{\Gamma(k+1)}\sum_{1\leqslant n\leqslant x}a_n(x-n)^k-R_k(x), \tag{3.3}$$

and assume that, for all positive k,

$$\Delta_k(x)=O\{x^{\eta+(1-\alpha)k}\}, \tag{3.4}$$

as $x\to\infty$, where $\quad 0<\alpha\leqslant 1, \qquad \eta<\rho. \quad$ (3.41)

Further, suppose that

$$\psi(s)=\sum_{n=1}^{\infty}a_n n^{-s}\quad(s=\sigma+it) \tag{3.5}$$

is convergent for sufficiently large values of σ, and can be continued analytically into the region $\sigma\geqslant\frac{1}{2}\beta$, where

$$\tfrac{1}{2}\beta<\theta, \tag{3.6}$$

and β is some real number.*

* The summation formulae discussed in (I) correspond to the case when $\eta<\frac{1}{2}\beta$.

Consider the function

$$\phi(x) = \Delta_k(x)x^{-\frac{1}{2}(\beta+1)-k}. \tag{3.7}$$

By (3.21) and (3.4)

$$\phi(x) = \begin{cases} O(x^{\theta-\frac{1}{2}\beta-\frac{1}{2}}) & (x \to 0), \\ O(x^{\eta-\frac{1}{2}\beta-\frac{1}{2}-\alpha k}) & (x \to \infty). \end{cases}$$

Hence, if

$$k > \frac{\eta-\frac{1}{2}\beta}{\alpha},$$

$\phi(x)$ belongs to $L^2(0,\infty)$. Let us take

$$k = \left[\frac{\eta-\frac{1}{2}\beta}{\alpha}\right]+1. \tag{3.8}$$

Then $\phi(x)$ has a Mellin transform* $\Phi(s)$ which belongs to $L^2(-\infty,\infty)$ on $\sigma = \frac{1}{2}$.

Further,

$$\Phi(s) = \int\limits_0^\infty \Delta_k(x)x^{s-\frac{1}{2}(\beta+3)-k}\,dx$$

since this integral converges absolutely for

$$\tfrac{1}{2}-(\theta-\tfrac{1}{2}\beta) < \sigma \leqslant \tfrac{1}{2}.$$

Hence

$$\Phi(s) = \int\limits_1^\infty \Delta_k(x)x^{s-\frac{1}{2}(\beta+3)-k}\,dx - \int\limits_0^1 R_k(x)x^{s-\frac{1}{2}(\beta+3)-k}\,dx$$

$$= \int\limits_1^\infty \Delta_k(x)x^{s-\frac{1}{2}(\beta+3)-k}\,dx - b\sum \int\limits_0^1 x^{r+s-\frac{1}{2}(\beta+3)-k}(\log x)^n\,dx$$

$$= \int\limits_1^\infty \Delta_k(x)x^{s-\frac{1}{2}(\beta+3)-k}\,dx - b\sum \frac{(-)^n n!}{(r+s-\frac{1}{2}\beta-\frac{1}{2}-k)^{n+1}}.$$

This obviously gives the analytic continuation of $\Phi(s)$ into the region $\sigma < \frac{1}{2}-(\theta-\frac{1}{2}\beta)$. If we take $\sigma < \frac{1}{2}-(\rho-\frac{1}{2}\beta)$, we have

$$\Phi(s) = \frac{1}{\Gamma(k+1)}\int\limits_1^\infty \Big\{\sum_{1\leqslant n\leqslant x} a_n(x-n)^k\Big\}x^{s-\frac{1}{2}(\beta+3)-k}\,dx -$$

$$- \int\limits_1^\infty R_k(x)x^{s-\frac{1}{2}(\beta+3)-k}\,dx - \sum \frac{b(-)^n n!}{(r+s-\frac{1}{2}\beta-\frac{1}{2}-k)^{n+1}},$$

since the second integral converges absolutely in this region.

* E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals* (Oxford, 1937), 94–5.

Evaluating the second integral as before, we find that it cancels with the last term, and we have

$$\Phi(s) = \frac{1}{\Gamma(k+1)} \int_1^\infty \Big\{ \sum_{1 \leqslant n \leqslant x} a_n (x-n)^k \Big\} x^{s-\frac{1}{2}(\beta+3)-k}\, dx$$

$$= \frac{1}{\Gamma(k)} \int_1^\infty x^{s-\frac{1}{2}(\beta+3)-k}\, dx \int_1^x (x-t)^{k-1} \Big\{ \sum_{1 \leqslant n \leqslant t} a_n \Big\}\, dt$$

$$= \frac{1}{\Gamma(k)} \int_1^\infty \Big\{ \sum_{1 \leqslant n \leqslant t} a_n \Big\}\, dt \int_t^\infty (x-t)^{k-1} x^{s-\frac{1}{2}(\beta+3)-k}\, dx$$

$$= \frac{\Gamma\{\frac{1}{2}(\beta+3)-s\}}{\Gamma\{\frac{1}{2}(\beta+3)+k-s\}} \sum_{n=1}^\infty (a_1+a_2+\ldots+a_n) \int_n^{n+1} t^{s-\frac{1}{2}(\beta+3)}\, dt$$

$$= \frac{\Gamma\{\frac{1}{2}(\beta+1)-s\}}{\Gamma\{\frac{1}{2}(\beta+3)+k-s\}} \sum_{n=1}^\infty (a_1+a_2+\ldots+a_n)\{n^{s-\frac{1}{2}(\beta+1)}-(n+1)^{s-\frac{1}{2}(\beta+1)}\}$$

$$= \frac{\Gamma\{\frac{1}{2}(\beta+1)-s\}}{\Gamma\{\frac{1}{2}(\beta+3)+k-s\}} \psi\{\tfrac{1}{2}(\beta+1)-s\}.$$

Now put
$$A(s) = \frac{\psi\{\frac{1}{2}(\beta-1)+s\}}{\psi\{\frac{1}{2}(\beta+1)-s\}}, \tag{3.9}$$

and $\chi_{1,\frac{1}{2}\beta+k+1}(x)$

$$= \underset{T\to\infty}{\mathrm{l.i.m.}} \frac{1}{2\pi i} \int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} \frac{\Gamma\{\frac{1}{2}(\beta+1)-s\}\Gamma\{\frac{1}{2}(\beta+1)+k+s\}}{\Gamma\{\frac{1}{2}(\beta+5)+k-s\}\Gamma\{\frac{1}{2}(\beta-1)+s\}} A(1-s) x^{1-s}\, ds. \tag{3.10}$$

Now
$$A(\tfrac{1}{2}-it)A(\tfrac{1}{2}+it) = 1, \qquad |A(\tfrac{1}{2}+it)| = 1,$$

and consequently, on the line $s = \frac{1}{2}+it$,

$$\left| \frac{\Gamma\{\frac{1}{2}(\beta+1)-s\}\Gamma\{\frac{1}{2}(\beta+1)+k+s\}}{\Gamma\{\frac{1}{2}(\beta+5)+k-s\}\Gamma\{\frac{1}{2}(\beta-1)+s\}} A(1-s) \right| = |\tfrac{1}{2}\beta+k+1-it|^{-1},$$

and thus belongs to $L^2(-\infty,\infty)$ when integrated with respect to t. Hence the integral (3.10) converges in mean square, and $\chi_{1,\frac{1}{2}\beta+k+1}(x)$ is a generalized Hankel kernel* of order $\frac{1}{2}\beta+k+1$.

* G. N. Watson, *Proc. London Math. Soc.* (2) 35 (1933), 156–99 (195).

Hence, by the Parseval theorem for Mellin transforms,

$$x^{\frac{1}{2}(\beta+1)+k}\int\limits_0^\infty \frac{\chi_{1,\frac{1}{2}\beta+k+1}(xy)}{y}\phi(y)\,dy$$

$$=\frac{x^{\frac{1}{2}(\beta+1)+k}}{2\pi i}\int\limits_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\frac{\Gamma\{\frac{1}{2}(\beta+1)-s\}\Gamma\{\frac{1}{2}(\beta+1)+k+s\}}{\Gamma\{\frac{1}{2}(\beta+5)+k-s\}\Gamma\{\frac{1}{2}(\beta-1)+s\}}A(1-s)x^{1-s}\times$$

$$\times\frac{\Gamma\{\frac{1}{2}(\beta-1)+s\}}{\Gamma\{\frac{1}{2}(\beta+1)+k+s\}}\psi\{\tfrac{1}{2}(\beta-1)+s\}\,ds$$

$$=\frac{x^{\frac{1}{2}(\beta+1)+k}}{2\pi i}\int\limits_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}\Phi(s)\frac{x^{1-s}}{\frac{1}{2}(\beta+3)+k-s}\,ds$$

$$=\int\limits_0^x \phi(y)y^{\frac{1}{2}(\beta+1)+k}\,dy.$$

Thus we have proved

THEOREM 1. *If* (i) *there is a sequence* $\{a_n\}$ *of real numbers and a real* β *such that the series* (3.5) *is convergent for sufficiently large values of* σ, *and* $\psi(s)$ *can be continued analytically into the region* $\sigma \geqslant \frac{1}{2}\beta$,

(ii) *there exists a function* $R_0(x)$ *which can be expressed as the sum of a finite number of terms of the form* $ax^m(\log x)^n$, *where* n *is a non-negative integer, and the greatest and least values of* m *are* ρ *and* θ *respectively; further,* $\Delta_k(x)$, *defined by* (3.1) *and* (3.3), *satisfies* (3.4), *and the inequalities* (3.41) *and* (3.6) *hold,*

then $$\int\limits_0^x \phi(y)y^{\frac{1}{2}(\beta+1)+k}\,dy = x^{\frac{1}{2}(\beta+1)+k}\int\limits_0^x \frac{\chi_{1,\frac{1}{2}\beta+k+1}(xy)}{y}\phi(y)\,dy,$$

where $\phi(x)$, k, *and* $\chi_{1,\frac{1}{2}\beta+k+1}(x)$ *are defined by* (3.7), (3.8), *and* (3.10). *Further,* $\chi_{1,\frac{1}{2}\beta+k+1}(x)$ *is a generalized Hankel kernel of order* $\frac{1}{2}\beta+k+1$.

4. The connexion between the two classes of transforms

Put $$\chi_{\frac{1}{2}\beta}(x) = \underset{T\to\infty}{\text{l.i.m.}}\frac{1}{2\pi i}\int\limits_{\frac{1}{2}-iT}^{\frac{1}{2}+iT}\frac{A(1-s)}{\frac{1}{2}(\beta+1)-s}x^{1-s}\,ds. \tag{4.1}$$

Then $\chi_{\frac{1}{2}\beta}(x)$ is a generalized Hankel kernel of order $\frac{1}{2}\beta$. Suppose that $f(x)$ is a function such that $f(x)$, $f'(x)$, $f''(x)$,..., $f^{(k)}(x)$ are integrals, and that $f(x)$, $xf'(x)$, $x^2f''(x)$,..., $x^{k+1}f^{(k+1)}(x)$ belong to $L^2(0,\infty)$.

Suppose further that $g(x)$ is the transform of $f(x)$ with respect to the generalized Hankel kernel $\chi_{\frac{1}{2}\beta}(x)$. That is,

$$\int_0^x g(y)y^{\frac{1}{2}(\beta-1)}\,dy = x^{\frac{1}{2}(\beta-1)}\int_0^\infty \frac{\chi_{\frac{1}{2}\beta}(xy)}{y}f(y)\,dy.$$

Then it can be shown by the method of Lemma α of (I) that $g(x)$ is almost everywhere differentiable, and may be chosen equal to the integral of its derivative, that $x^{\frac{1}{2}}f(x)$ and $x^{\frac{1}{2}}g(x)$ tend to zero as x tends to zero or infinity, that $g(x)$ and $xg'(x)$ belong to $L^2(0,\infty)$, and that

$$x^{\frac{1}{2}(\beta+1)}\frac{d}{dx}\{x^{-\frac{1}{2}(\beta-1)}f(x)\}, \qquad x^{\frac{1}{2}(\beta+1)}\frac{d}{dx}\{x^{-\frac{1}{2}(\beta-1)}g(x)\}$$

are transforms with respect to the generalized Hankel kernel

$$\underset{T\to\infty}{\text{l.i.m.}}\ \frac{1}{2\pi i}\int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT}\frac{\frac{1}{2}(\beta-1)+s}{\{\frac{1}{2}(\beta+3)-s\}\{\frac{1}{2}(\beta+1)-s\}}A(1-s)x^{1-s}\,ds$$

of order $\frac{1}{2}\beta+1$. Applying this process $k+1$ times we find

Lemma α. *If $f(x), f'(x), f''(x),\ldots, f^{(k)}(x)$ are integrals, and $f(x)$, $xf'(x)$, $x^2f''(x),\ldots, x^{k+1}f^{(k+1)}(x)$ belong to $L^2(0,\infty)$, then $f(x)$ has a transform $g(x)$ with respect to the generalized Hankel kernel $\chi_{\frac{1}{2}\beta}(x)$. Further $g(x)$, $g'(x),\ldots, g^{(k)}(x)$ exist and can be chosen equal to the integrals of their derivatives, and $g(x)$, $xg'(x)$, $x^2g''(x),\ldots, x^{k+1}g^{(k+1)}(x)$ belong to $L^2(0,\infty)$. Also, for $0 \leqslant r \leqslant k$, $x^{r+\frac{1}{2}}f^{(r)}(x)$ and $x^{r+\frac{1}{2}}g^{(r)}(x)$ approach zero as x approaches zero or infinity, and*

$$x^{\frac{1}{2}(\beta+1)+k}\left(\frac{d}{dx}\right)^{k+1}\{x^{-\frac{1}{2}(\beta-1)}f(x)\}, \qquad x^{\frac{1}{2}(\beta+1)+k}\left(\frac{d}{dx}\right)^{k+1}\{x^{-\frac{1}{2}(\beta-1)}g(x)\}$$

are transforms of $L^2(0,\infty)$ with respect to the Hankel kernel $\chi_{1,\frac{1}{2}\beta+k+1}(x)$.

5. The summation formula

With the assumptions of Theorem 1 and Lemma α, Parseval's theorem for generalized Hankel transforms gives

$$\int_0^\infty \Delta_k(x)\left(\frac{d}{dx}\right)^{k+1}\{x^{-\frac{1}{2}(\beta-1)}f(x)\}\,dx = \int_0^\infty \Delta_k(x)\left(\frac{d}{dx}\right)^{k+1}\{x^{-\frac{1}{2}(\beta-1)}g(x)\}\,dx.$$

The left-hand side is

$$\int\limits_N^\infty \Delta_k(x)\left(\frac{d}{dx}\right)^{k-1}\{x^{-\frac{1}{2}(\beta-1)}f(x)\}\,dx+$$

$$+(-)^k\left\{\left[\sum_{m=0}^{k}(-)^m\Delta_m(x)\left(\frac{d}{dx}\right)^m\{x^{-\frac{1}{2}(\beta-1)}f(x)\}\right]_0^N-\right.$$

$$\left.-\sum_{n=1}^{N}a_n\,n^{-\frac{1}{2}(\beta-1)}f(n)+\int\limits_0^N x^{-\frac{1}{2}(\beta-1)}f(x)\,dR_0(x)\right\},$$

on integrating by parts $k+1$ times as in § 2. By (3.21) and Lemma α the integrated terms vanish at the lower limit. At the upper limit the term $m=k$ is $O(N^{\eta-\frac{1}{2}\beta-\alpha k})$ and by (3.8) this is $o(1)$. Hence we have

THEOREM 2. *If the conditions of Theorem* 1 *and Lemma* α *are satisfied, then*

$$\lim_{N\to\infty}\left\{\sum_{n=1}^{N}a_n\,n^{-\frac{1}{2}(\beta-1)}f(n)-\int\limits_0^N x^{-\frac{1}{2}(\beta-1)}f(x)\,dR_0(x)-\right.$$

$$\left.-\sum_{m=0}^{k-1}(-)^m\Delta_m(N)\left[\left(\frac{d}{dx}\right)^m\{x^{-\frac{1}{2}(\beta-1)}f(x)\}\right]_{x=N}\right\}$$

$$=\lim_{N\to\infty}\left\{\sum_{n=1}^{N}a_n\,n^{-\frac{1}{2}(\beta-1)}g(n)-\int\limits_0^N x^{-\frac{1}{2}(\beta-1)}g(x)\,dR_0(x)-\right.$$

$$\left.-\sum_{m=0}^{k-1}(-)^m\Delta_m(N)\left[\left(\frac{d}{dx}\right)^m\{x^{-\frac{1}{2}(\beta-1)}g(x)\}\right]_{x=N}\right\}. \qquad (5.1)$$

6. Summability by Riesz means

Let us write

$$S(N)=\sum_{n=1}^{N}a_n\,n^{-\frac{1}{2}(\beta-1)}f(n)-\int\limits_0^N x^{-\frac{1}{2}(\beta-1)}f(x)\,dR_0(x)$$

$$=(-)^k\left[\int\limits_0^\infty-\int\limits_N^\infty\Delta_k(x)\left(\frac{d}{dx}\right)^{k+1}\{x^{-\frac{1}{2}(\beta-1)}f(x)\}\,dx\right]+$$

$$+\sum_{m=0}^{k}(-)^m\Delta_m(N)\left[\left(\frac{d}{dx}\right)^m\{x^{-\frac{1}{2}(\beta-1)}f(x)\}\right]_{x=N} \qquad (6.1)$$

and consider the problem of the limitability of $S(N)$ by Riesz means (R, n, κ) as N tends to infinity. We have to consider

$$\lim_{X\to\infty}\left\{\kappa X^{-\kappa}\int_0^X (X-t)^{\kappa-1}S(t)\,dt\right\}. \tag{6.2}$$

Obviously the first integral in the second part of (6.1) remains unchanged, being independent of N, and the second integral and the term $m = k$ contribute nothing to (6.2) since they both vanish as N tends to infinity, and we have to consider

$$\lim_{X\to\infty}\left[\kappa X^{-\kappa}\sum_{m=0}^{k-1}(-)^m\int_0^X (X-t)^{\kappa-1}\Delta_m(t)\left(\frac{d}{dt}\right)^m\{t^{-\frac{1}{2}(\beta-1)}f(t)\}\,dt\right]. \tag{6.3}$$

Put $$F(t) = t^{-\frac{1}{2}(\beta-1)}f(t)$$

and consider $$X^{-\kappa}\int_0^X (X-t)^{\kappa-1}\Delta_m(t)F^{(m)}(t)\,dt. \tag{6.4}$$

By Lemma α $$F^{(m)}(t) = o(t^{-\frac{1}{2}\beta-m}) \tag{6.5}$$

at the origin and infinity. Let us assume that this also holds for values of m up to $\kappa+k-2$, and that κ is an integer. Integrating (6.4) by parts $\kappa-1$ times, the integrated terms vanish, and it becomes

$$(-)^{\kappa-1}X^{-\kappa}\int_0^X \Delta_{m+\kappa-1}(t)\left(\frac{d}{dt}\right)^{\kappa-1}\{(X-t)^{\kappa-1}F^{(m)}(t)\}\,dt.$$

This may be written as the sum of a finite series of terms of the form

$$A_r X^{-\kappa}\int_0^X \Delta_{m+\kappa-1}(t)(X-t)^r F^{(m+r)}(t)\,dt \quad (0 \leqslant r \leqslant \kappa-1).$$

By (3.21) and (6.5) the integral over the range $(0, \Delta)$ contributes a term $$O(X^{r-\kappa}) = O(X^{-1}).$$

By (3.4) and (6.5) the integral over the range (Δ, X) is

$$O\left\{X^{-\kappa}\int_\Delta^X t^{\eta-\frac{1}{2}\beta+\alpha+\kappa-\alpha(m+\kappa)-r-1}(X-t)^r\,dt\right\}$$

$$= O\left\{X^{\eta-\frac{1}{2}\beta+\alpha-\alpha(\kappa+m)}\int_{\Delta/X}^1 u^{\eta-\frac{1}{2}\beta+\alpha+\kappa-\alpha(m+\kappa)-r-1}(1-u)^r\,du\right\}.$$

If
$$\eta-\tfrac{1}{2}\beta+\alpha+\kappa-\alpha(m+\kappa)-r < 0,$$
this is
$$O\{X^{r-\kappa}\} = O(X^{-1}).$$
If
$$\eta-\tfrac{1}{2}\beta+\alpha+\kappa-\alpha(m+\kappa)-r = 0,$$
it is
$$O(X^{r-\kappa}\log X) = O(X^{-1}\log X);$$
and, if
$$\eta-\tfrac{1}{2}\beta+\alpha+\kappa-\alpha(m+\kappa)-r > 0,$$
it is
$$O\{X^{\eta-\frac{1}{2}\beta+\alpha-\alpha(\kappa+m)}\} = O\{X^{\eta-\frac{1}{2}\beta+\alpha-\alpha\kappa}\} = o(1)$$
if
$$\kappa > \frac{\eta-\frac{1}{2}\beta}{\alpha}+1.$$
Hence, taking
$$\kappa = \left[\frac{\eta-\frac{1}{2}\beta}{\alpha}\right]+2 = k+1, \tag{6.6}$$
it follows that (6.3) vanishes.

In order to ensure that (6.5) holds, it will be sufficient to assume that $f(x)$ satisfies the conditions of Lemma α with $k+\kappa-2$ instead of k. Then $g(x)$ will satisfy similar conditions, and, substituting the first part of (6.1) in (6.2), we have

THEOREM 3. *If the assumptions* (i) *and* (ii) *of Theorem* 1 *are satisfied, and*

(iii) *$f(x)$, $f'(x)$, $f''(x)$,..., $f^{(2k-1)}(x)$ are integrals, and $f(x)$, $xf'(x)$, $x^2f''(x)$,..., $x^{2k}f^{(2k)}(x)$ belong to $L^2(0,\infty)$, then*

$$\lim_{N\to\infty}\left\{\sum_{n=1}^{N} a_n\left(1-\frac{n}{N}\right)^{k+1} n^{-\frac{1}{2}(\beta-1)}f(n)-\int_0^N \left(1-\frac{x}{N}\right)^{k+1} x^{-\frac{1}{2}(\beta-1)}f(x)\,dR_0(x)\right\}$$
$$=\lim_{N\to\infty}\left\{\sum_{n=1}^{N} a_n\left(1-\frac{n}{N}\right)^{k+1} n^{-\frac{1}{2}(\beta-1)}g(n)-\int_0^N \left(1-\frac{x}{N}\right)^{k+1} x^{-\frac{1}{2}(\beta-1)}g(x)\,dR_0(x)\right\},$$

where
$$\int_0^x g(y)y^{\frac{1}{2}(\beta-1)}\,dy = x^{\frac{1}{2}(\beta-1)}\int_0^\infty \frac{\chi_{\frac{1}{2}\beta}(xy)}{y}f(y)\,dy,$$

$\chi_{\frac{1}{2}\beta}(x)$ is defined by (4.1), *and $g(x)$ is chosen to be the integral of its derivative.*

7. Discontinuous functions

The formula (1.2) corresponds to

$$f(t) = \begin{cases} t^{\frac{1}{2}p+\frac{1}{2}} & (t<x), \\ \frac{1}{2}x^{\frac{1}{2}p+\frac{1}{2}} & (t=x), \\ 0 & (t>x), \end{cases}$$

and this case is not covered by Theorem 3. As in (I) we can extend our results to cover such cases. The process we use is fairly straightforward when applied to any of the known examples, but requires a number of additional conditions in the general case.

Let us write
$$F(x) = \Delta_0(x)x^{-\frac{1}{2}(\beta+1)-\frac{1}{2}l},$$
$$G(x) = \Delta_l(x)x^{-\frac{1}{2}(\beta+1)-\frac{1}{2}l},$$
and suppose that l is an integer satisfying
$$\eta-\tfrac{1}{2}\beta < \tfrac{1}{2}l < \theta-\tfrac{1}{2}\beta,$$
$$\eta-\tfrac{1}{2}\beta < (\alpha-\tfrac{1}{2})l, \tag{7.1}$$
assuming that the inequality (3.4) holds for $0 \leqslant k \leqslant l$.* Then $F(x)$ and $G(x)$ belong to $L^2(0,\infty)$, and it can be shown by the method of § 3 that their Mellin transforms are
$$\mathfrak{F}(s) = \frac{\psi(\frac{1}{2}\beta+\frac{1}{2}l+\frac{1}{2}-s)}{\frac{1}{2}\beta+\frac{1}{2}l+\frac{1}{2}-s}$$
and
$$\mathfrak{G}(s) = \frac{\Gamma(\frac{1}{2}\beta-\frac{1}{2}l+\frac{1}{2}-s)}{\Gamma(\frac{1}{2}\beta+\frac{1}{2}l+\frac{3}{2}-s)}\psi(\tfrac{1}{2}\beta-\tfrac{1}{2}l+\tfrac{1}{2}-s).$$
Hence, if we put
$$\mathfrak{H}_1(s) = A(1-s+\tfrac{1}{2}l)\frac{\{\frac{1}{2}(\beta-1)-\frac{1}{2}l+s\}\Gamma\{\frac{1}{2}(\beta+1)+\frac{1}{2}l+s\}}{\{\frac{1}{2}(\beta+1)+\frac{1}{2}l-s\}\Gamma\{\frac{1}{2}(\beta+1)-\frac{1}{2}l+s\}},$$
$$\mathfrak{H}_2(s) = A(1-s-\tfrac{1}{2}l)\frac{\{\frac{1}{2}(\beta-1)+\frac{1}{2}l+s\}\Gamma\{\frac{1}{2}(\beta+3)-\frac{1}{2}l-s\}}{\{\frac{1}{2}(\beta+1)-\frac{1}{2}l-s\}\Gamma\{\frac{1}{2}(\beta+3)+\frac{1}{2}l-s\}},$$
then
$$\mathfrak{H}_1(s)\mathfrak{H}_2(1-s) = 1, \qquad \mathfrak{F}(s) = \mathfrak{H}_1(s)\mathfrak{G}(1-s).$$
Assuming that $|A(\frac{1}{2}+\frac{1}{2}l+it)|$ is bounded for all t, and
$$|A(\tfrac{1}{2}+\tfrac{1}{2}k+it)| = O(t^{-k}) \tag{7.2}$$
for $k = \pm l$, it follows that $|\mathfrak{H}_1(\frac{1}{2}+it)|$ and $|\mathfrak{H}_2(\frac{1}{2}+it)|$ are bounded, and hence that they give rise to an unsymmetrical inversion formula of the Fourier type.†

Further, let
$$\omega_1(x) = \underset{T\to\infty}{\text{l.i.m.}}\frac{1}{2\pi i}\int\limits_{\frac{1}{2}-iT}^{\frac{1}{2}+iT}\frac{\mathfrak{H}_1(s)}{1-s}x^{1-s}\,ds,$$

* It is possible to find such an integer l in the cases which arise in practice. If it were not so, we would have to use fractional integrals and the theorem for fractional integration by parts.

† E. C. Titchmarsh, *An Introduction to the Theory of Fourier Integrals* (Oxford, 1937), 226.

and define similarly $\omega_2(x)$. Then, by the method of § 2,

$$\int_0^x F(y)\,dy = \int_0^\infty \frac{\omega_1(xy)}{y} G(y)\,dy.$$

We now require the following lemma, which is the extension to unsymmetrical inversion formulae of Lemma β of (I), and is proved in the same way.

LEMMA β. *If* (i) $\omega_1(x)$ *and* $\omega_2(x)$ *are integrals and have almost everywhere derivatives* $H_1(x)$ *and* $H_2(x)$ *which belong to* L^2 *over any finite range,*

(ii) $$B(x,y,\lambda) = \int_0^\lambda H_1(ux)H_2(uy)\,du$$

and $(x-y)B(x,y,\lambda)$ *is bounded for all positive* x, y, λ,

(iii) $$\lim_{\lambda\to\infty} \int_0^\lambda H_1(ua)\omega_2(ub)\frac{du}{u} = \begin{cases} 1 & (0<a<b), \\ \frac{1}{2} & (a=b), \\ 0 & (0<b<a), \end{cases}$$

(iv) $$\int_{y-\delta}^{y+\delta} |B(x,y,\lambda)|\,dx < A$$

for some positive δ *and all* y *and* λ,

(v) $f(x)$ *belongs to* $L^2(0,\infty)$ *and is of bounded variation in some neighbourhood of* $x = y$,

then $$\tfrac{1}{2}\{f(y+0)+f(y-0)\} = \lim_{\lambda\to\infty} \int_0^\lambda g(x)H_1(xy)\,dx,$$

where $g(x)$ *is the* ω_2*-transform of* $f(x)$.

With these conditions we have

$$\begin{aligned}\tfrac{1}{2}\{F(x+0)+F(x-0)\} &= \lim_{N\to\infty} \int_0^N H_1(xt)G(t)\,dt \\ &= \lim_{N\to\infty} \int_0^N H_1(xt)t^{-\frac{1}{2}(\beta+1)-\frac{1}{2}l}\Delta_l(t)\,dt. \qquad (7.3)\end{aligned}$$

If we integrate by parts l times and proceed as in § 4, we find that

$$\begin{aligned}\sum_{1\leqslant n\leqslant x}{}' a_n - R_0(x) = \lim_{N\to\infty} \Bigg\{ &\sum_{r=0}^{l} \left(\frac{x}{N}\right)^{\frac{1}{2}\beta+\frac{1}{2}r} L_r(Nx)\,\Delta_r(N) + \\ &+ \sum_{n=1}^{N} a_n n^{-\beta} H(nx) - \int_0^N H(xy)y^{-\beta}\,dR_0(y)\Bigg\}, \qquad (7.4)\end{aligned}$$

where

$$L_r(z) = -\frac{z^{\frac{1}{2}(l-r+1)}}{2\pi i}\times$$

$$\times \int\limits_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{\{\frac{1}{2}(\beta-1)-\frac{1}{2}l+s\}\Gamma\{\frac{1}{2}(\beta-1)+r-\frac{1}{2}l+s\}}{\{\frac{1}{2}(\beta+1)+\frac{1}{2}l-s\}\Gamma\{\frac{1}{2}(\beta+1)-\frac{1}{2}l+s\}} A(1-s+\tfrac{1}{2}l)z^{-s}\,ds, \tag{7.5}$$

and $$H(x) = -z^{\frac{1}{2}\beta}L_0(z) = \frac{z^{\frac{1}{2}(\beta+1)}}{2\pi i}\int\limits_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{A(1-s)}{\frac{1}{2}(\beta+1)-s} z^{-s}\,ds. \tag{7.6}$$

From (7.2) and (7.5) it follows that

$$L_r(z) = O(z^{\frac{1}{2}l-\frac{1}{2}r}),$$

and, applying the method of § 6 to (7.4), we find

THEOREM 4. *If, in addition to the assumptions of Theorem* 1 *and Lemma* β,

(i) *there is an integer* l *satisfying the inequalities* (7.1),

(ii) (7.2) *holds for all values of* k *in* $-l \leqslant k \leqslant l$, *then*

$$\sum_{1\leqslant n\leqslant x}{}' a_n - R_0(x) = \lim_{N\to\infty}\left\{\sum_{n=1}^{N} a_n\left(1-\frac{n}{N}\right)^{\lambda} n^{-\beta}H(nx) - \right.$$

$$\left. - \int\limits_0^N \left(1-\frac{x}{N}\right)^{\lambda} H(xy)y^{-\beta}\,dR_0(y)\right\},$$

where $$\lambda = \left[\frac{\eta-\frac{1}{2}\beta+\frac{1}{2}l}{\alpha}\right]+2. \tag{7.7}$$

8. Examples

(A) $$a_n = r_p(n), \qquad p \geqslant 4, \qquad \beta = \tfrac{1}{2}p.$$

It has been shown that, with the notation of § 2,*

$$P_k(x) = \begin{cases} O\{x^{\frac{1}{2}p-1+k/(p-1)+\epsilon}\} & (0 \leqslant k \leqslant \frac{1}{2}p-\frac{1}{2}), \quad (8.1)\\ O\{x^{\frac{1}{2}p-\frac{1}{2}+\frac{1}{2}k}\} & (k > \frac{1}{2}p-\frac{1}{2}). \quad (8.2)\end{cases}$$

Hence it will be sufficient if we take

$$\eta = \tfrac{1}{2}p-1+\epsilon, \qquad \alpha = \tfrac{1}{2},$$

and hence $$k = [\tfrac{1}{2}p]-1.$$

* J. R. Wilton, loc. cit.

We have

THEOREM 5.* *If $f(x)$ satisfies condition* (iii) *of Theorem* 3 *with* $k = [\frac{1}{2}p]-1$, $p \geqslant 4$, *then*

$$\lim_{N\to\infty}\left\{\sum_{n=1}^{N}\left(1-\frac{n}{N}\right)^{k+1} r_p(n)n^{-\frac{1}{4}p+\frac{1}{4}}f(n)-\frac{\pi^{\frac{1}{2}p}}{\Gamma(\frac{1}{2}p)}\int_0^N\left(1-\frac{x}{N}\right)^{k+1} x^{\frac{1}{4}p-\frac{1}{4}}f(x)\,dx\right\}$$

$$=\lim_{N\to\infty}\left\{\sum_{n=1}^{N}\left(1-\frac{n}{N}\right)^{k+1} r_p(n)n^{-\frac{1}{4}p+\frac{1}{4}}g(n)-\right.$$

$$\left.-\frac{\pi^{\frac{1}{2}p}}{\Gamma(\frac{1}{2}p)}\int_0^N\left(1-\frac{x}{N}\right)^{k+1} x^{\frac{1}{4}p-\frac{1}{4}}g(x)\,dx\right\},$$

where $$\int_0^x g(y)y^{\frac{1}{4}p-\frac{1}{4}}\,dy = x^{\frac{1}{4}p}\int_0^\infty y^{-\frac{1}{4}}J_{\frac{1}{2}p}(2\pi x^{\frac{1}{2}}y^{\frac{1}{2}})f(y)\,dy,$$

and $g(x)$ is chosen so that it is the integral of its derivative.

(B) If we apply the method of § 7 to the above case (7.3) becomes

$$x^{-\frac{1}{4}p-\frac{1}{4}-\frac{1}{2}l}\left\{\sum_{1\leqslant n\leqslant x}' r_p(n)-\frac{\pi^{\frac{1}{2}p}x^{\frac{1}{2}p}}{\Gamma(1+\frac{1}{2}p)}\right\}$$

$$=\pi^{1+l}\int_0^\infty P_l(t)t^{-\frac{1}{4}p-\frac{1}{4}-\frac{1}{2}l}J_{\frac{1}{2}p+l+1}(2\pi x^{\frac{1}{2}}t^{\frac{1}{2}})\,dt,$$

where $l = [\frac{1}{2}p-\frac{1}{2}]$, and we can deduce by partial integration that

$$\sum_{1\leqslant n\leqslant x}' r_p(n)-\frac{\pi^{\frac{1}{2}p}x^{\frac{1}{2}p}}{\Gamma(1+\frac{1}{2}p)} = \sum_{n=1}^{N} r_p(n)\left(\frac{x}{n}\right)^{\frac{1}{2}p} J_{\frac{1}{2}p}(2\pi n^{\frac{1}{2}}x^{\frac{1}{2}})+$$

$$+\pi^{\lambda}\int_N^\infty\left(\frac{x}{t}\right)^{\frac{1}{4}p+\frac{1}{2}\lambda} J_{\frac{1}{2}p+\lambda}(2\pi x^{\frac{1}{2}}t^{\frac{1}{2}})P_{\lambda-1}(t)\,dt -$$

$$-\sum_{r=0}^{\lambda-1}\pi^{r}\left(\frac{x}{N}\right)^{\frac{1}{4}p+\frac{1}{2}r} J_{\frac{1}{2}p+r}(2\pi N^{\frac{1}{2}}x^{\frac{1}{2}})P_r(N) +$$

$$+\sum_{r=1}^{[\frac{1}{2}p]}\frac{\pi^{\frac{1}{2}p-r}}{\Gamma(\frac{1}{2}p-r+1)}N^{\frac{1}{2}p-r}\left(\frac{x}{N}\right)^{\frac{1}{4}p-\frac{1}{2}r} J_{\frac{1}{2}p-r}(2\pi N^{\frac{1}{2}}x^{\frac{1}{2}})+\theta_p(x),$$

where $$\lambda = [\tfrac{1}{2}p-\tfrac{1}{2}],$$

and $$\theta_p(x) = \frac{2}{\pi}\int_{2\pi N^{\frac{1}{2}}x^{\frac{1}{2}}}^\infty \frac{\sin t}{t}\,dt$$

if p is odd, and zero if p is even.

* Obviously we could prove this result with a slightly lower value of k if we used the two inequalities (8.1) and (8.2) in § 6 instead of combining them in one of the form (3.4).

This result has been given by Wilton (loc. cit.), and he has shown that the formula (1.2) can be deduced from it.

(C) $$a_n = \sigma_r(n), \qquad \beta = r+1 \quad (r \neq 0).$$

It has been shown that, if $0 \leqslant k \leqslant |r|+\frac{1}{2}$, $q = \max(|r|, 1)$, then*

$$\Delta_k(x) = O\{x^{\frac{1}{2}q+\frac{1}{2}r+k(|r|+1-q)/(2|r|+1)+\epsilon}\},$$

and, if $$k > |r|+\tfrac{1}{2},$$

then $$\Delta_k(x) = O\{x^{\frac{1}{2}k+\frac{1}{2}r+\frac{1}{4}}\}.$$

Hence it will be sufficient if we take

$$\eta = \tfrac{1}{2}(q+r)+\epsilon, \qquad \alpha = \tfrac{1}{2},$$

and hence $k = [q]$. We also have

$$R_0(x) = x^{r+1}\zeta(1+r)/(1+r)+x\zeta(1-r).$$

Hence $\rho = 1+r$, $\theta = 1$ if $r > 0$, and $\rho = 1$, $\theta = 1+r$ if $r < 0$. Thus the conditions of Theorem 3 are satisfied only for $|r| < 1$.†

We have

THEOREM 6. *If $f(x)$, $f'(x)$ are integrals, $f(x)$, $xf'(x)$, $x^2f''(x)$ belong to $L^2(0,\infty)$, and $0 < |r| < 1$, then*

$$\lim_{N\to\infty}\left\{\sum_{n=1}^{N}\left(1-\frac{n}{N}\right)^2\sigma_r(n)n^{-\frac{1}{2}r}f(n)- -\int_0^N\left(1-\frac{x}{N}\right)^2 f(x)[x^{-\frac{1}{2}r}\zeta(1-r)+x^{\frac{1}{2}r}\zeta(1+r)]\,dx\right\}$$

$$=\lim_{N\to\infty}\left\{\sum_{n=1}^{N}\left(1-\frac{n}{N}\right)^2\sigma_r(n)n^{-\frac{1}{2}r}g(n)- -\int_0^N\left(1-\frac{x}{N}\right)^2 g(x)[x^{-\frac{1}{2}r}\zeta(1-r)+x^{\frac{1}{2}r}\zeta(1+r)]\,dx\right\},$$

where $$\int_0^x g(y)y^{\frac{1}{2}r}\,dy = x^{\frac{1}{2}r+\frac{1}{2}}\int_0^\infty y^{-\frac{1}{2}}f(y)F_{r+1}(4\pi x^{\frac{1}{2}}y^{\frac{1}{2}})\,dy,$$

$$F_\nu(z) = \cos\tfrac{1}{2}\pi\nu J_\nu(z)-\sin\tfrac{1}{2}\pi\nu\left\{Y_\nu(z)+\frac{2}{\pi}K_\nu(z)\right\},$$

and $g(x)$ is chosen so that it is the integral of its derivative.

* J. R. Wilton, *Proc. London Math. Soc.* (2) 36 (1933), 391–426.

† If we used the theory of transforms of functions of $L^p(0,\infty)$ $(1 < p \leqslant 2)$ throughout the proof of Theorem 3 we could extend the result to $|r| < 2$. The modifications necessary in Theorem 1 if $|r| \geqslant 2$ have been given by J. R. Wilton (loc. cit., Theorems A and A′), but considerably more stringent conditions would have to be imposed on $f(x)$ in order to justify the result of Theorem 3 in this case.

MINKOWSKI'S INEQUALITY FOR THE MINIMA ASSOCIATED WITH A CONVEX BODY

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LET K be an open convex domain in n-dimensional space, with the origin O as centre (that is, symmetrical about O). The classical theorem of Minkowski asserts that, if the volume $V(K)$ of K is greater than 2^n, then K contains a lattice-point (i.e. a point with integral coordinates) other than O. The various proofs of this theorem which have been given are all based on the observation that any domain (whether convex or not) which has volume greater than 2^n must contain two different points $\mathbf{x} = (x_1,..., x_n)$, $\mathbf{y} = (y_1,..., y_n)$ such that*

$$x_1 \equiv y_1 \pmod 2,..., \qquad x_n \equiv y_n \pmod 2. \tag{1}$$

Intuition suggests that this will be the case, and a number of proofs have been given.† If $\mathbf{x}$, $\mathbf{y}$ are two such points in K, then $\frac{1}{2}(\mathbf{x}-\mathbf{y})$ is a lattice-point other than O in K.

Minkowski also proved the following more general theorem.‡ For any $\lambda > 0$, let λK denote the domain consisting of all points $\lambda\mathbf{x}$, where $\mathbf{x}$ is a point of K. Let λ_1 be the least number for which the domain $\lambda_1 K$ has a lattice-point, say P_1, on its boundary. Let λ_2 be the least number for which $\lambda_2 K$ has a lattice-point, say P_2, not collinear with O, P_1 on its boundary. Let λ_3 be the least number for which $\lambda_3 K$ has a lattice-point, say P_3, not coplanar with O, P_1, P_2 on its boundary, and so on. This process defines n numbers $\lambda_1,..., \lambda_n$ satisfying $0 < \lambda_1 \leqslant \lambda_2 \leqslant ... \leqslant \lambda_n$. The more general theorem of Minkowski asserts that

$$\lambda_1 \lambda_2 ... \lambda_n V(K) \leqslant 2^n. \tag{2}$$

The previous theorem is equivalent to $V(\lambda_1 K) \leqslant 2^n$, or $\lambda_1^n V(K) \leqslant 2^n$, and so is contained in (2).

The object of this note is to give a simple proof of (2).

* Two real numbers a, b satisfy $a \equiv b \pmod 2$ if $a = b+2m$, where m is an integer.

† For one proof see Koksma, *Diophantische Approximationen* (Berlin, 1936), Kapitel II, § 1, Satz 4. This proof was given independently (in slightly different forms) by Hajós and Mordell.

‡ *Geometrie der Zahlen* (Berlin, 1910), Kapitel V.

Define points $Q_1, Q_2,..., Q_n$ as follows. Q_1 is P_1; Q_2 is any lattice-point such that Q_1 and Q_2 generate all lattice-points in the plane OP_1P_2; Q_3 is any lattice-point such that Q_1, Q_2, Q_3 generate all lattice-points in the linear space $OP_1P_2P_3$, and so on. The linear transformation of the n-dimensional space which transforms Q_1 into $(1, 0, 0,..., 0)$, Q_2 into $(0, 1, 0,..., 0)$,..., Q_n into $(0, 0, 0,..., 1)$ transforms the lattice into itself. Hence it is an integral unimodular transformation, and changes any convex domain with centre O into another with the same volume. We suppose this transformation carried out.

By definition, $\lambda_n K$ is an open convex domain with one lattice-point (or more), not in the linear space $OP_1...P_{n-1}$, on its boundary, but no such lattice-point inside it. Thus any lattice-point in $\lambda_n K$ has $x_n = 0$. Any lattice-point in $\lambda_{n-1}K$ has $x_n = x_{n-1} = 0$, and so on generally. If $\mathbf{x}$, $\mathbf{y}$ are two points in $\lambda_r K$ satisfying (1), then

$$x_r = y_r, \qquad x_{r+1} = y_{r+1}, ..., \qquad x_n = y_n, \tag{3}$$

since $\frac{1}{2}(\mathbf{x}-\mathbf{y})$ is a lattice-point in $\lambda_r K$.

We shall suppose that

$$\lambda_1 \lambda_2 ... \lambda_n V(K) > 2^n, \tag{4}$$

and deduce a contradiction. It is sufficient to construct domains $K_1, K_2,..., K_n$, not necessarily convex, such that

(*a*) K_r is contained in $\lambda_r K$;

(*b*) if $\mathbf{x}$, $\mathbf{y}$ are points of K_r and $x_i = y_i$ for $i = r, r+1,..., n$, then there exist points $\mathbf{x}'$, $\mathbf{y}'$ in K_{r-1} such that $\mathbf{x}-\mathbf{y} = \mathbf{x}'-\mathbf{y}'$;

(*c*) $V(K_n) = \lambda_1...\lambda_n V(K)$.

For then, by (4) and (*c*), there exist two different points $\mathbf{x}$, $\mathbf{y}$ in K_n satisfying (1). Since K_n is contained in $\lambda_n K$, this implies $x_n = y_n$, by (3). Hence, by (*b*), there exist two points $\mathbf{x}'$, $\mathbf{y}'$ in K_{n-1} satisfying (1). Since K_{n-1} is contained in $\lambda_{n-1}K$, this implies $x'_n = y'_n$, $x'_{n-1} = y'_{n-1}$ by (3). Continuing the argument, we obtain two (different) points in K_1 satisfying (1), and hence a lattice-point other than O in $\lambda_1 K$, which is impossible.

The construction is as follows. We define K_1 to be $\lambda_1 K$. We suppose that K_{r-1} has been defined, and proceed to define K_r.

Let $\phi_1(x_r,..., x_n),..., \phi_{r-1}(x_r,..., x_n)$ be any functions such that $(\phi_1,..., \phi_{r-1}, x_r,..., x_n)$ lies in $\lambda_{r-1}K$. Such functions can be defined for any point $(x_r,..., x_n)$ in the projection of $\lambda_{r-1}K$ upon the space

$x_1 = ... = x_{r-1} = 0$, so as to be continuous functions of $x_r,..., x_n$ in this projection. We define K_r to be the set of all points $\mathbf{X}$ given by

$$\begin{aligned}
X_1 &= x_1+\left(\frac{\lambda_r}{\lambda_{r-1}}-1\right)\phi_1(x_r,..., x_n),\\
X_2 &= x_2+\left(\frac{\lambda_r}{\lambda_{r-1}}-1\right)\phi_2(x_r,..., x_n),\\
&\cdots\cdots\cdots\cdots\\
X_{r-1} &= x_{r-1}+\left(\frac{\lambda_r}{\lambda_{r-1}}-1\right)\phi_{r-1}(x_r,..., x_n),\\
X_r &= \frac{\lambda_r}{\lambda_{r-1}}x_r,\\
&\cdots\cdots\cdots\cdots\\
X_n &= \frac{\lambda_r}{\lambda_{r-1}}x_n,
\end{aligned}$$

when $\mathbf{x}$ runs through all points of K_{r-1}. Since $(x_1,..., x_n)$ and $(\phi_1,..., \phi_{r-1}, x_r,..., x_n)$ lie in $\lambda_{r-1} K$, it follows from the convexity of K that $\mathbf{X}$ lies in $\lambda_r K$. Hence (a) holds for K_r if it holds for K_{r-1}. It is obvious that (b) holds. Finally,

$$V(K_r) = \left(\frac{\lambda_r}{\lambda_{r-1}}\right)^{n-r+1} V(K_{r-1}),$$

whence

$$V(K_n) = \lambda_1 ... \lambda_n V(K).$$

THE MEAN VALUE OF THE ZETA-FUNCTION ON THE CRITICAL LINE

By F. V. ATKINSON (*Oxford*)

[Received 13 December 1938]

THE object of this paper is to prove the following formulae:

THEOREM I. *As* $T \to \infty$,

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^2\,dt = T\log T - T(1+\log 2\pi - 2\gamma) + O(T^{\frac{1}{2}}\log^2 T).$$

THEOREM II. *As* $\delta \to 0$,

$$\int_0^\infty |\zeta(\tfrac{1}{2}+it)|^2 e^{-\delta t}\,dt$$
$$= \frac{1}{\delta}\log\frac{1}{\delta} - \frac{1}{\delta}(\log 2\pi - \gamma) + \sum_0^{N-1} \delta^n(a_n + b_n \log\delta) + O(\delta^N \log\delta),$$

where the a_n, b_n are certain constants and N is any positive integer.

Neither of these results is new. In each case the dominant term on the right-hand side is due to Hardy and Littlewood (**1**). Ingham (**2**) has proved Theorem I with an error term of $O(T^{\frac{1}{2}}\log T)$, and Titchmarsh (**5**) has improved this to $O(T^{\frac{5}{12}}\log^2 T)$. Theorem II for $N = 0$ is given by Wilton (**10**), and the general case is due to Kober (**3**), whose expansion, however, contains unnecessary terms in $\log^2\delta$. The proofs here given are slightly more elementary than those of the above writers.

I am indebted to Professor Titchmarsh for suggesting this problem to me, and for much advice as regards the preparation of this paper.

Proof of Theorem I.

We assume the following results on $\zeta(s)$, proved in Titchmarsh (**6**). The functional equation,

$$\zeta(1-s) = \chi(1-s)\zeta(s) = 2^{1-s}\pi^{-s}\cos\tfrac{1}{2}s\pi\,\Gamma(s)\zeta(s); \tag{1.1}$$

$$\zeta(\sigma+it) = \begin{cases} O(t^{\frac{1}{2}(1-\sigma)}\log t), & \text{uniformly for } 0 \leqslant \sigma \leqslant 1; \quad (1.2) \\ O(\log t), & \text{uniformly for } \sigma \geqslant 1. \quad (1.3) \end{cases}$$

Then, by (1.2), $\qquad |\zeta(\tfrac{1}{2}+it)|^2 = O(t^{\frac{1}{2}}\log^2 t);$

therefore, without loss of generality, we may write $T/2\pi = N+\frac{1}{2}$, where N is a positive integer. Let us write λ for $1/\log T$, and let C

denote the contour formed by the three lines joining the points $\frac{1}{2}-iT$, $1+\lambda-iT$, $1+\lambda+iT$, $\frac{1}{2}+iT$. Then we have

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^2\,dt = \frac{1}{2i}\int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} \zeta(s)\zeta(1-s)\,ds$$

$$= \frac{1}{2i}\int_C \zeta(s)\zeta(1-s)\,ds+\tfrac{1}{2}\pi = \frac{1}{2i}\int_C \chi(1-s)\zeta^2(s)\,ds+\tfrac{1}{2}\pi$$

$$= \frac{1}{2i}\int_C \chi(1-s)\sum_{n\leqslant N}\frac{d(n)}{n^s}\,ds+\frac{1}{2i}\int_C \chi(1-s)\Big\{\zeta^2(s)-\sum_{n\leqslant N}\frac{d(n)}{n^s}\Big\}\,ds+\tfrac{1}{2}\pi$$

$$= \sum_{n\leqslant N}\frac{d(n)}{2i}\int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT}\chi(1-s)n^{-s}\,ds+$$

$$+\frac{1}{2i}\left\{\int_{\frac{1}{2}-iT}^{1+\lambda-iT}+\int_{1+\lambda+iT}^{\frac{1}{2}+iT}\right\}\chi(1-s)\Big\{\zeta^2(s)-\sum_{n\leqslant N}\frac{d(n)}{n^s}\Big\}\,ds+$$

$$+\sum_{n>N}\frac{d(n)}{2i}\int_{1+\lambda-iT}^{1+\lambda+iT}\chi(1-s)n^{-s}\,ds+\tfrac{1}{2}\pi,\text{ since }\zeta^2(s)=\sum_1^\infty\frac{d(n)}{n^s}\text{ for }\sigma>1,$$

$$= \sum_{n\leqslant N}d(n)I_n+\frac{1}{2i}(J_1+J_2)+\sum_{n>N}d(n)K_n+\tfrac{1}{2}\pi,\text{ say.}$$

We shall now prove the following lemmas.

LEMMA 1. *For $n\leqslant N$,*

$$I_n = 2\pi+O\left(\frac{1}{n^{\frac{1}{2}}\log(T/2\pi n)}\right),\quad\textit{and}\quad I_n = 2\pi+O(T^{\frac{1}{2}}n^{-\frac{1}{2}}).$$

For

$$I_n = \frac{1}{i}\left\{\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty}-\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}-iT}-\int_{\frac{1}{2}+iT}^{\frac{1}{2}+i\infty}\right\}\Gamma(s)\cos\tfrac{1}{2}s\pi(2\pi n)^{-s}\,ds = A_1-A_2-A_3,\text{ say.}$$

Now $A_1 = 2\pi\cos 2\pi n = 2\pi$. To deal with A_3 we observe that, by Stirling's formula,

$$\Gamma(\sigma+it)$$
$$= t^{\sigma-\frac{1}{2}}e^{-\frac{1}{2}\pi t}\sqrt{(2\pi)}\exp i\Big\{t\log t-t+\tfrac{1}{2}\pi(\sigma-\tfrac{1}{2})+\frac{\sigma-\sigma^2-\frac{1}{6}}{2t}+O(t^{-2})\Big\}.$$

Therefore

$$A_3 = n^{-\frac{1}{2}}\int_T^\infty \exp i\Big\{t\log t-t-t\log(2\pi n)+\frac{1}{24t}\Big\}\{1+O(t^{-2})\}\,dt$$

multiplied by a constant.

Now the term $O(t^{-2})$ contributes

$$O\left(n^{-\frac{1}{2}}\int_{T}^{\infty} t^{-2}\,dt\right) = O(n^{-\frac{1}{2}}\,T^{-1}).$$

If we write $f(t)$ for the function inside the brackets $\{\}$, we have

$$f'(t) = \log\frac{t}{2\pi n}-\frac{1}{24t^2} > c_1\log\frac{T}{2\pi n}$$

for $t > T$ and some fixed $c_1 > 0$. Also

$$f''(t) = \frac{1}{t}+\frac{1}{12t^3} > \frac{c_2}{T}$$

for $t < 2T$ and some fixed $c_2 > 0$.

It follows by some well-known theorems on integrals, proved in Titchmarsh (7), (8), that

$$\int_{T}^{\infty} \exp i\{f(t)\}\,dt = O\left(\frac{1}{\log(T/2\pi n)}\right),$$

and also $$= O(T^{\frac{1}{2}}).$$

Therefore

$$A_3 = O\left(\frac{1}{n^{\frac{1}{2}}\log(T/2\pi n)}\right)+O(n^{-\frac{1}{2}}T^{-1}) = O\left(\frac{1}{n^{\frac{1}{2}}\log(T/2\pi n)}\right),$$

and also

$$= O(T^{\frac{1}{2}}n^{-\frac{1}{2}})+O(n^{-\frac{1}{2}}T^{-1}) = O(T^{\frac{1}{2}}n^{-\frac{1}{2}}).$$

A_2 may be dealt with similarly and the lemma is therefore proved.

LEMMA 2.

$$J_1 = O(T^{\frac{1}{2}}\log^2 T), \qquad J_2 = O(T^{\frac{1}{2}}\log^2 T).$$

We have

$$J_2 = \frac{1}{2i}\int_{1+\lambda+iT}^{\frac{1}{2}+iT} \chi(1-s)\zeta^2(s)\,ds - \frac{1}{2i}\int_{1+\lambda+iT}^{\frac{1}{2}+iT} \chi(1-s)\sum_{n\leqslant N}\frac{d(n)}{n^s}\,ds$$

$$= j_1-j_2, \text{ say.}$$

But, by Stirling's formula,

$$|\chi(1-s)| = O(t^{\sigma-\frac{1}{2}}), \quad \text{for } t > h > 0,$$

and so, by (1.2) and (1.3),

$$|\chi(1-s)\zeta^2(s)| = O(t^{\frac{1}{2}}\log^2 t) \quad \text{uniformly for } \tfrac{1}{2} \leqslant \sigma \leqslant 1+\lambda.$$

Therefore $$j_1 = O\left(\int_{\frac{1}{2}}^{1+\lambda} T^{\frac{1}{2}}\log^2 T\,d\sigma\right) = O(T^{\frac{1}{2}}\log^2 T).$$

Now $$\sum_{n<x} d(n) = x\log x + x(2\gamma-1) + O(x^{\frac{1}{2}}).$$

Therefore, by partial summation,

$$\sum_{n\leqslant N} \frac{d(n)}{n^\sigma} = \begin{cases} O\left(\dfrac{T^{1-\sigma}\log T}{1-\sigma}\right) & (\sigma < 1), \\ O(T\log^2 T) & (1-\lambda \leqslant \sigma \leqslant 1+\lambda). \end{cases}$$

Hence

$$j_2 = O\left(\int\limits_{\frac{1}{2}}^{1-\lambda} T^{\sigma-\frac{1}{2}} \frac{T^{1-\sigma}\log T}{1-\sigma}\, d\sigma\right) + O\left(\int\limits_{1-\lambda}^{1+\lambda} T^{\sigma-\frac{1}{2}} \log^2 T\, d\sigma\right)$$

$$= O(T^{\frac{1}{2}}\log^2 T).$$

The result for J_2 follows; similarly that for J_1.

Lemma 3. *For $n > N$,*

$$K_n = O\left(\frac{T^{\frac{1}{2}}}{n^{1+\lambda}\log(T/2\pi n)}\right) + O(n^{-1-\lambda})$$

and also $$= O(T\, n^{-1-\lambda}).$$

For

$$K_n = \frac{1}{i}\int\limits_{1+\lambda-i}^{1+\lambda+i} + \int\limits_{1+\lambda+i}^{1+\lambda+iT} + \int\limits_{1+\lambda-iT}^{1+\lambda-i} \Gamma(s)\cos\tfrac{1}{2}s\pi(2\pi n)^{-s}\, ds = B_1 + B_2 + B_3, \text{ say.}$$

Now plainly $$B_1 = O(n^{-1-\lambda}).$$

The discussion of B_2 is practically the same as that of A_3, except that we have an extra factor $t^{\frac{1}{2}+\lambda}$ in the integrand. Hence we obtain

$$B_2 = O\left(\frac{T^{\frac{1}{2}+\lambda}}{n^{1+\lambda}\log(T/2\pi n)}\right) = O\left(\frac{T^{\frac{1}{2}}}{n^{1+\lambda}\log(T/2\pi n)}\right),$$

and also $$B_2 = O(T^{1+\lambda}n^{-1-\lambda}) = O(T\, n^{-1-\lambda}).$$

Similarly, we may prove the same inequalities for B_3, and so the lemma is proved.

Applying our lemmas, we have

$$\int\limits_0^T |\zeta(\tfrac{1}{2}+it)|^2\, dt = 2\pi \sum_{n\leqslant N} d(n) + \sum_{n<N-\sqrt{N}} d(n) O\left(\frac{1}{n^{\frac{1}{2}}\log(T/2\pi n)}\right) +$$

$$+ \sum_{N-\sqrt{N}\leqslant n\leqslant N} d(n) O(T^{\frac{1}{2}} n^{-\frac{1}{2}}) + \sum_{N<n\leqslant N+\sqrt{N}} d(n) O(T n^{-1-\lambda}) +$$

$$+ \sum_{N+\sqrt{N}<n} d(n) O\left(\frac{T^{\frac{1}{2}}}{n^{1+\lambda}\log(T/2\pi n)}\right) + O(T^{\frac{1}{2}}\log^2 T)$$

$$= \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4 + \Sigma_5 + O(T^{\frac{1}{2}}\log^2 T).$$

Now

$$\sum_1 = 2\pi \sum_{n<(T/2\pi)} d(n) = 2\pi\left(\frac{T}{2\pi}\log\frac{T}{2\pi}+(2\gamma-1)\frac{T}{2\pi}\right)+O(T^{\frac{1}{2}})$$

$$= T\log T - T(1+\log 2\pi - 2\gamma)+O(T^{\frac{1}{2}});$$

$$\sum_2 = \sum_{n\leqslant\frac{1}{2}N} + \sum_{\frac{1}{2}N<n<N-\surd N} d(n)O\left(\frac{1}{n^{\frac{1}{4}}\log(T/2\pi n)}\right) = \sum_2' + \sum_2''.$$

Again $\log(T/2\pi n) > \log 2$, when $n < T/4\pi$, and so

$$\sum_2' = O\left(\sum_{n<\frac{1}{2}N} \frac{d(n)}{n^{\frac{1}{4}}}\right) = O(T^{\frac{3}{4}}\log T).$$

For $\frac{1}{2}N < n < N-\surd N$,

$$n^{\frac{1}{4}} = O(T^{\frac{1}{4}}), \qquad \log\frac{T}{2\pi n} > \frac{T-2\pi n}{T}.$$

Therefore

$$\sum_2'' = O\left(T^{\frac{3}{4}} \sum_{\frac{1}{2}N<n<N-\surd N} \frac{d(n)}{T-2\pi n}\right)$$

$$= O(T^{\frac{3}{4}}\log^2 T), \quad \text{by partial summation.}$$

$$\sum_3 = \sum_{N-\surd N\leqslant n\leqslant N} d(n)O\left(\frac{T^{\frac{1}{4}}}{n^{\frac{1}{4}}}\right) = O\left(\sum_{N-\surd N\leqslant n\leqslant N} d(n)\right) = O(T^{\frac{1}{2}}\log T);$$

$$\sum_4 = \sum_{N<n\leqslant N+\surd N} d(n)O\left(\frac{T^{1+\lambda}}{n^{1+\lambda}}\right) = O\left(\sum_{N<n\leqslant N+\surd N} d(n)\right) = O(T^{\frac{1}{2}}\log T);$$

$$\sum_5 = \sum_{N+\surd N<n<2N} + \sum_{n>2N} d(n)O\left(\frac{T^{\frac{3}{4}}}{n^{1+\lambda}\log(T/2\pi n)}\right) = \sum_5' + \sum_5''.$$

As in the case of $\sum_2''$ we get

$$\sum_5' = O\left(T^{\frac{3}{4}} \sum_{N+\surd N<n<2N} \frac{d(n)}{2\pi n - T}\right) = O(T^{\frac{3}{4}}\log^2 T), \text{ by partial summation.}$$

For $n > T/\pi$, $\log(2\pi n/T) > \log 2$, and so

$$\sum_5'' = O\left(T^{\frac{3}{4}} \sum_{n\geqslant 2N} \frac{d(n)}{n^{1+\lambda}}\right) = O(T^{\frac{3}{4}}\log^2 T), \text{ by partial summation.}$$

On combining these results, Theorem I follows.

Proof of Theorem II.

We show first that it is sufficient to prove this theorem for a slightly different integral. Write

$$\phi(\delta) = \int_0^\infty |\zeta(\tfrac{1}{2}+it)|^2 e^{-\delta t}\,dt - \frac{e^{-\frac{1}{2}i\delta}}{i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{\zeta(s)\zeta(1-s)}{2\cos\frac{1}{2}s\pi} e^{-is(\frac{1}{2}\pi-\delta)}\,ds$$

$$= \int_0^\infty |\zeta(\tfrac{1}{2}+it)|^2 e^{-\delta t}\,dt - \int_{-\infty}^{\infty} \frac{|\zeta(\frac{1}{2}+it)|^2 e^{-\frac{1}{4}i\pi+\frac{1}{2}\pi t}e^{-\delta t}}{e^{-\frac{1}{4}i\pi+\frac{1}{2}\pi t}+e^{\frac{1}{4}i\pi-\frac{1}{2}\pi t}}\,dt$$

$$= \int_0^\infty \frac{|\zeta(\frac{1}{2}+it)|^2 e^{-\delta t}e^{\frac{1}{4}i\pi-\frac{1}{2}\pi t}}{e^{-\frac{1}{4}i\pi+\frac{1}{2}\pi t}+e^{\frac{1}{4}i\pi-\frac{1}{2}\pi t}}\,dt + \int_0^\infty \frac{|\zeta(\frac{1}{2}+it)|^2 e^{\delta t}e^{-\frac{1}{4}i\pi-\frac{1}{2}\pi t}}{e^{-\frac{1}{4}i\pi-\frac{1}{2}\pi t}+e^{\frac{1}{4}i\pi+\frac{1}{2}\pi t}}\,dt.$$

Now both these integrals are uniformly convergent for $|\delta| < \theta < \pi$. It follows that $\phi(\delta)$ is analytic in the region $|\delta| < \pi$. Now

$$\frac{e^{-\frac{1}{2}i\delta}}{2i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{\zeta(s)\zeta(1-s)}{\cos\frac{1}{2}s\pi} e^{-is(\frac{1}{2}\pi-\delta)}\,ds = \frac{e^{-\frac{1}{2}i\delta}}{i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \Gamma(s)(2\pi i e^{-i\delta})^{-s}\zeta^2(s)\,ds$$

$$= ie^{\frac{1}{2}i\delta}\{i(\tfrac{1}{2}\pi-\delta)+\log 2\pi-\gamma\}+\frac{e^{-\frac{1}{2}i\delta}}{i}\int_{2-i\infty}^{2+i\infty} \Gamma(s)(2\pi i e^{-i\delta})^{-s}\zeta^2(s)\,ds$$

$$= ie^{\frac{1}{2}i\delta}\{i(\tfrac{1}{2}\pi-\delta)+\log 2\pi-\gamma\}+2\pi e^{-\frac{1}{2}i\delta}\sum_1^\infty d(n)\exp(-2n\pi i e^{-i\delta}).$$

We have therefore

$$\int_0^\infty |\zeta(\tfrac{1}{2}+it)|^2 e^{-\delta t}\,dt = 2\pi e^{-\frac{1}{2}i\delta}\sum_1^\infty d(n)\exp(-2n\pi i e^{-i\delta})+\psi(\delta),$$

where $\psi(\delta)$ is analytic in the region $|\delta| < \pi$.

Now it has been shown by Wigert (9), and by Landau (4) by considering $\int \Gamma(s)\zeta^2(s)z^{-s}\,ds$, that

$$\sum_1^\infty d(n)e^{-nz} = \frac{\gamma-\log z}{z}+\tfrac{1}{4}+\sum_{r=0}^{k-1} c_r z^{2r+1}+O(z^{2k}),$$

when $z \to 0$, $-\alpha < \arg z < \alpha$, $0 < \alpha < \frac{1}{2}\pi$, and the c_r are certain constants.

In this put $z = 2\pi i(e^{-i\delta}-1)$. We get

$$\sum_1^\infty d(n)\exp\{-2\pi ni(e^{-i\delta}-1)\} = \sum_1^\infty d(n)\exp(-2\pi nie^{-i\delta})$$

$$= \frac{\gamma-\log\{2\pi i(e^{-i\delta}-1)\}}{2\pi i(e^{-i\delta}-1)}+\tfrac{1}{4}+\sum_{r=0}^{k-1} c_r\{2\pi i(e^{-i\delta}-1)\}^{2r+1}+O\{(e^{-i\delta}-1)^{2k}\}.$$

Hence

$$2\pi e^{-\frac{1}{2}i\delta}\sum_1^\infty d(n)\exp(-2\pi nie^{-i\delta})$$

$$= \frac{\gamma-\log\{2\pi\delta(1-\frac{1}{2}i\delta+...)\}}{2\sin\frac{1}{2}\delta}+\tfrac{1}{2}\pi e^{-\frac{1}{2}i\delta}+$$

$$+\sum_{r=0}^{k-1} 2\pi c_r\, e^{-\frac{1}{2}i\delta}\{2\pi i(e^{-i\delta}-1)\}^{2r+1}+O(\delta^{2k})$$

$$= \frac{\gamma-\log(2\pi\delta)}{2\sin\frac{1}{2}\delta}+\sum_{r=0}^{2k-1} d_r\,\delta^r+O(\delta^{2k}), \quad \text{for } |\delta|<\pi, \text{ as } \delta\to 0,$$

$$= \frac{1}{\delta}\log\frac{1}{\delta}+\frac{\gamma-\log 2\pi}{\delta}+\sum_{r=0}^{2k-1}(f_r+g_r\log\delta)\delta^r+O(\delta^{2k}),$$

where the d_r, f_r, g_r denote certain constants.

But we have shown that

$$\psi(\delta) = \int_0^\infty |\zeta(\tfrac{1}{2}+it)|^2 e^{-\delta t}\,dt - 2\pi e^{-\frac{1}{2}i\delta}\sum_1^\infty d(n)\exp(-2\pi nie^{-i\delta})$$

is analytic in the region $|\delta|<\pi$, and may therefore be expanded in a power series in δ convergent inside this region. This proves Theorem II.

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NOTE ON THE TRANSFORMATION OF EULERIAN HYPERGEOMETRIC INTEGRALS

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1. It was shown by Riemann that the hypergeometric series $F(\alpha,\beta;\gamma;z)$ can be represented by forty-eight different Eulerian integrals, which can be divided into two classes, each class containing twenty-four integral representations. Assuming

$$\Re(\gamma) > \Re(\alpha) > 0 \quad \text{and} \quad \Re(\gamma) > \Re(\beta) > 0, \tag{1}$$

the integrals of the first class can be reduced by an elementary change of the variable of integration to

$$\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int\limits_0^1 x^{\beta-1}(1-x)^{\gamma-\beta-1}(1-xz)^{-\alpha}\,dx, \tag{2}$$

and those of the second class quite similarly to

$$\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\int\limits_0^1 x^{\alpha-1}(1-x)^{\gamma-\alpha-1}(1-xz)^{-\beta}\,dx. \tag{3}$$

The two integrals (2) and (3), however, cannot in general be directly transformed into each other. The question arises, how can (2) be transformed in (3) and vice versa.

Riemann writes on this subject in his famous essay, 'Beiträge zur Theorie der durch die Gaussche Reihe $F(\alpha,\beta,\gamma,x)$ darstellbaren Funktionen'* the following lines: 'Die übrigen Gleichungen [e.g. the identity of (2) and (3)] erfordern, soweit ich sie untersucht habe, zu ihrer Bestätigung durch Methoden der Integralrechnung die Transformation von vielfachen Integralen.' After an unsuccessful attempt of Schellenberg,† Wirtinger‡ transformed two Eulerian integrals belonging to different classes into each other, evaluating a *triple* integral in two different manners. Later on I succeeded§ in the transformation of (2) into (3) by means of a *double* integral.

* B. Riemann's *Gesammelte mathematische Werke und wissenschaftlicher Nachlass* (Leipzig, 1876), 62–78.

† Schellenberg, *Dissertation* (Göttingen, 1892).

‡ W. Wirtinger, *Sitzungsber. Akad. Wiss. Wien*, 111 (1902), 894–900.

§ A. Erdélyi, *Quart. J. of Math.* (Oxford), 8 (1937), 267–77. Especially 272–3.

Recently Poole* pointed out that the two classes of Eulerian integrals can be directly transformed into each other by integrations by parts, when the hypergeometric equation has a logarithmic singularity, i.e. when any of the differences of exponents is an integer. Reading Poole's paper, I conjectured that his method is generalizable by using fractional derivatives or integrals, and by extending the conception of integration by parts to fractional derivatives. In this note I shall explain this generalization of Poole's method. The fractional integration by parts is nearly related to transformation by double integrals—thus confirming Riemann's conjecture—and therefore does not yield any essentially new transformation. Nevertheless, it is to a certain extent interesting to see how Poole's method is connected with the transformation of Eulerian integrals by means of double integrals. At the same time my method also exhibits the transformation by means of double integrals from a new point of view.

The assumption (1) I shall maintain for the sake of simplicity throughout. This is no essential restriction, since it can easily be removed by using contour integrals instead of (2) and (3).

2. I begin by recalling Poole's method of transformation of (2) into (3). $F(\alpha,\beta;\gamma;z)$ belongs to the Riemannian scheme

$$P\left\{\begin{matrix} 0 & 1 & \infty & \\ 0 & 0 & \alpha & z \\ 1-\gamma & \gamma-\alpha-\beta & \beta & \end{matrix}\right\}.$$

A logarithmic singularity occurs, in general, when the difference of the two exponents belonging to one of the singularities happens to be an integer. Owing to the fact that permutation of the singularities does not alter the character of Riemann's P-function at all, we can assume $z = \infty$ to be the logarithmic singularity, that is to say, $\beta-\alpha$ to be a positive integer.

Now we can transform (2) integrating by parts. Then (2) is obviously equal to

$$\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\alpha)}\int\limits_0^1 x^{\beta-1}(1-xz)^{-\alpha}\frac{d^{\beta-\alpha}(1-x)^{\gamma-\alpha-1}}{d(-x)^{\beta-\alpha}}\,dx.$$

* E. G. C. Poole, *Quart. J. of Math.* (Oxford), 9 (1938), 230–3.

Integrating $\beta-\alpha$ times by parts, the integrated terms cancel for $x = 0$ and $x = 1$, leaving only

$$\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\alpha)}\int_0^1 (1-x)^{\gamma-\alpha-1}\frac{d^{\beta-\alpha}x^{\beta-1}(1-xz)^{-\alpha}}{dx^{\beta-\alpha}}\,dx. \tag{4}$$

Now,

$$\frac{d^{\beta-\alpha}x^{\beta-1}(1-xz)^{-\alpha}}{dx^{\beta-\alpha}} = \frac{d^{\beta-\alpha}}{dx^{\beta-\alpha}}\sum_{r=0}^{\infty}\frac{\Gamma(\alpha+r)x^{\beta+r-1}z^r}{\Gamma(\alpha)\Gamma(r+1)}$$

$$= \sum_{r=0}^{\infty}\frac{\Gamma(\beta+r)x^{\alpha+r-1}z^r}{\Gamma(\alpha)\Gamma(r+1)} = \frac{\Gamma(\beta)}{\Gamma(\alpha)}x^{\alpha-1}(1-xz)^{-\beta}.$$

Putting this result in (4), we obtain

$$\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\int_0^1 x^{\alpha-1}(1-x)^{\gamma-\alpha-1}(1-xz)^{-\beta}\,dx.$$

Thus we have obtained (3) by a direct transformation of (2).

3. The rule for ordinary repeated integration by parts runs

$$\int_a^b u\frac{d^n v}{d(-x)^n}\,dx = \int_a^b v\frac{d^n u}{dx^n}\,dx,$$

provided that the integrated terms cancel. This is, for instance, true if u and its first $n-1$ derivatives vanish for $x = a$, and v and its first $n-1$ derivatives vanish for $x = b$.

The fractional derivatives we define by the formal relation

$$\frac{d^\nu x^\kappa}{dx^\nu} = \frac{\Gamma(\kappa+1)x^{\kappa-\nu}}{\Gamma(\kappa-\nu+1)}. \tag{5}$$

Assuming for u and v respectively the expansions

$$u = \sum_{r=0}^{\infty} A_r(x-a)^{\rho+r-1}, \qquad v = \sum_{s=0}^{\infty} B_s(b-x)^{\sigma+s-1},$$

and taking the definition (5) for fractional derivatives, we at once obtain the straightforward generalization of integration by parts, namely,

$$\int_a^b u\frac{d^\nu v}{d(-x)^\nu}\,dx = \int_a^b v\frac{d^\nu u}{dx^\nu}\,dx. \tag{6}$$

In (6) ν is any real or complex number such that the fractional derivatives exist and the integrals converge.

To prove this rule, we remark that

$$\frac{d^\nu v}{d(-x)^\nu} = \frac{d^\nu v}{d(b-x)^\nu} = \sum_{s=0}^{\infty} \frac{\Gamma(\sigma+s)B_s}{\Gamma(\sigma+s-\nu)}(b-x)^{\sigma+s-\nu-1},$$

and therefore

$$\int_a^b u\frac{d^\nu v}{d(-x)^\nu}\,dx = \int_a^b \sum_{r=0}^{\infty} A_r(x-a)^{\rho+r-1} \sum_{s=0}^{\infty} \frac{\Gamma(\sigma+s)B_s}{\Gamma(\sigma+s-\nu)}(b-x)^{\sigma+s-\nu-1}\,dx$$

$$= \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{A_r B_s \Gamma(\sigma+s)}{\Gamma(\sigma+s-\nu)} \int_a^b (x-a)^{\rho+r-1}(b-x)^{\sigma+s-\nu-1}\,dx$$

$$= \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} A_r B_s \frac{\Gamma(\rho+r)\Gamma(\sigma+s)}{\Gamma(\rho+\sigma-\nu+r+s)}(b-a)^{\rho+\sigma-\nu+r+s-1},$$

and that a similar computation of the right of (6) yields the same result.

With the definition (5) *of fractional derivatives and the rule* (6) *for fractional integration by parts, Poole's transformation as described in* § 2 *formally holds for any real or complex value of* $\beta-\alpha$.

4. Fractional integration by parts can also be used to obtain a more general transformation of (2), yielding a functional equation of the hypergeometric series. To obtain this transformation we integrate (2) by parts $(\beta-\lambda)$ times, assuming $\Re(\gamma) > \Re(\lambda) > 0$. Thus we obtain

$$\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^1 x^{\beta-1}(1-x)^{\gamma-\beta-1}(1-xz)^{-\alpha}\,dx$$

$$= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\lambda)}\int_0^1 x^{\beta-1}(1-xz)^{-\alpha}\frac{d^{\beta-\lambda}(1-x)^{\gamma-\lambda-1}}{d(-x)^{\beta-\lambda}}\,dx$$

$$= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\lambda)}\int_0^1 (1-x)^{\gamma-\lambda-1}\frac{d^{\beta-\lambda}x^{\beta-1}(1-xz)^{-\alpha}}{dx^{\beta-\lambda}}\,dx.$$

Now

$$\frac{d^{\beta-\lambda}x^{\beta-1}(1-xz)^{-\alpha}}{dx^{\beta-\lambda}} = \frac{d^{\beta-\lambda}}{dx^{\beta-\lambda}}\sum_{r=0}^{\infty}\frac{\Gamma(\alpha+r)x^{\beta+r-1}z^r}{\Gamma(\alpha)\Gamma(r+1)}$$

$$= \sum_{r=0}^{\infty}\frac{\Gamma(\alpha+r)\Gamma(\beta+r)x^{\lambda+r-1}z^r}{\Gamma(\alpha)\Gamma(\lambda+r)\Gamma(r+1)}$$

$$= \frac{\Gamma(\beta)}{\Gamma(\lambda)}x^{\lambda-1}F(\alpha,\beta;\lambda;xz),$$

and therefore

$$\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int\limits_0^1 x^{\beta-1}(1-x)^{\gamma-\beta-1}(1-xz)^{-\alpha}\,dx$$
$$=\frac{\Gamma(\gamma)}{\Gamma(\lambda)\Gamma(\gamma-\lambda)}\int\limits_0^1 x^{\lambda-1}(1-x)^{\gamma-\lambda-1}F(\alpha,\beta;\lambda;xz)\,dx.$$

We thus have the functional equation

$$F(\alpha,\beta;\gamma;z)=\frac{\Gamma(\gamma)}{\Gamma(\lambda)\Gamma(\gamma-\lambda)}\int\limits_0^1 x^{\lambda-1}(1-x)^{\gamma-\lambda-1}F(\alpha,\beta;\lambda;xz)\,dx.$$

This functional equation is, as I have pointed out in an earlier paper,* another means of connecting Eulerian integrals of different classes, (2) and (3) being special cases of this functional equation. In a subsequent article I shall prove a generalization of this integral formula, by the same method.

The reader will immediately see that the methods dealt with in this paper also apply to some related problems in the theory of generalized hypergeometric series.† Similar methods I hope to apply in a subsequent paper to the Laplace integrals representing confluent hypergeometric functions.

5. I conclude by bringing out the connexion between the transformation of integrals by integration by parts and the transformation by double integrals. To simplify the matter and to avoid contour integrals I put $\nu=-s$ $[\Re(s)>0]$.

Instead of the definition of fractional derivatives given in § 3 we can use the equivalent definitions of fractional *integrals*:

$$\frac{d^{-s}u}{dx^{-s}}=\mathop{\mathrm{I}}_a^x{}^s\,u=\frac{1}{\Gamma(s)}\int\limits_a^x(x-y)^{s-1}u(y)\,dy, \tag{7}$$

and
$$\frac{d^{-s}v}{d(-x)^{-s}}=\mathop{\mathrm{I}}_x^b{}^s\,v=\frac{1}{\Gamma(s)}\int\limits_x^b(y-x)^{s-1}v(y)\,dy. \tag{8}$$

* A. Erdélyi, *Quart. J. of Math.* (Oxford), 8 (1937), 200–13, p. 203.

† These problems are treated in my paper in *Quart. J. of Math.* (Oxford), 8 (1937), 267–77.

The rule for fractional integration by parts, (6), with these notations, runs

$$\int_a^b u \mathop{\mathrm{I}}_x^b{}^s v\,dx = \int_a^b v \mathop{\mathrm{I}}_a^x{}^s u\,dx. \tag{9}$$

This form of the rule for fractional integration by parts is more general than (6), since, using the definitions (7), (8) for fractional integrals, no expansibility of u, v into power series is required.

Now (9) implies that

$$\int_a^b u(x)\left[\int_x^b (y-x)^{s-1}v(y)\,dy\right]dx = \int_a^b v(x)\left[\int_a^x (x-y)^{s-1}u(y)\,dy\right]dx,$$

and this is true, both sides being equal to the double integral

$$\iint u(x)v(y)(y-x)^{s-1}\,dxdy$$

taken over the domain $a \leqslant x \leqslant y \leqslant b$. Thus we see that fractional integration by parts is equivalent to transformation of integrals by means of double integrals. So, too, is ordinary integration by parts. The formula

$$\int_a^b uv'\,dx = -\int_a^b u'v\,dx + [uv]_a^b$$

is based on the evaluation of the double integral

$$\iint u'(x)v'(y)\,dxdy,$$

taken over a suitable domain, in two different ways.

NOTE ON A PAPER BY WINTNER

By L. A. PARS (*Cambridge*)

[Received 10 January 1939]

WE consider a dynamical system for which

$$T = \tfrac{1}{2}a_{rs}\dot{q}_r\dot{q}_s,$$

where the coefficients a_{rs} are homogeneous functions of $q_1, q_2,..., q_n$, of degree $-\alpha$, and the work function U is homogeneous of degree β. Wintner has shown* that, if $h = 0$,

$$q_r a_{rs}\dot{q}_s = (\beta-\alpha+2)\int U\,dt.$$

An alternative proof may be of interest. If we multiply the rth Lagrangian equation by q_r, and sum for $r = 1$ to $r = n$, we get

$$q_r\frac{d}{dt}(a_{rs}\dot{q}_s)-\tfrac{1}{2}q_r\frac{\partial a_{ij}}{\partial q_r}\dot{q}_i\dot{q}_j = q_r\frac{\partial U}{\partial q_r},$$

and, by using Euler's theorem on homogeneous functions, this becomes, if we add $2T$ to each side,

$$q_r\frac{d}{dt}(a_{rs}\dot{q}_s)+a_{rs}\dot{q}_r\dot{q}_s+\alpha T = \beta U+2T,$$

i.e.

$$\frac{d}{dt}(q_r a_{rs}\dot{q}_s) = \beta U-(\alpha-2)T.$$

Hence, since $T = h+U$ for all t,

$$\frac{d}{dt}(q_r a_{rs}\dot{q}_s) = (\beta-\alpha+2)U-(\alpha-2)h.$$

Wintner's theorem is the special case of this when $h = 0$.

* A. Wintner, *Quart. J. of Math.* (Oxford), 7 (1936), 214–18.

A TYPE OF AEROFOIL

By S. D. DAYMOND (*Liverpool*) and J. HODGKINSON (*Oxford*)

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THE problem of two-dimensional irrotational flow past an aerofoil with a sharp trailing edge has often been discussed.* The usual method of treatment is to employ a conformal transformation which converts the space outside a circle into the space outside an elongated closed curve determined by the transformation, the curve having the general characteristics of an aerofoil section, the trailing edge being sharp, the camber small, and the chord (the distance from the trailing edge to the centre of curvature of the forward rounded edge) large compared with the maximum thickness. The resultant force acting on the aerofoil is then determined by standard methods.

In this paper, by supposing the contour of the aerofoil section to consist of three circular arcs touching one another in pairs, we are able to specify in advance the camber and the maximum thickness of the aerofoil. The necessary conformal representation is effected by means of the complete elliptic integrals, and we obtain the velocity potential we require as the real part of a function which is perfectly general, whatever the camber and the thickness of the aerofoil section. Our later calculations are made on the assumption that the maximum thickness is small in comparison with the length of the chord.

We find that, when the angle between the chord and the direction of motion of the aerofoil lies between 5° and 20°, the line of action of the resultant force meets the chord at a point whose distance from the leading edge is approximately constant, and about one-third the length of the chord. This is a practical result which has been established by experiment.†

1. The figure representing the aerofoil section in the z-plane (Fig. 1) is composed of arcs of three circles which touch one another in pairs at the points O, A, Ω. Three independent quantities are necessary to specify the size of the figure; the orientation of the figure is such that the y-axis is the common tangent at O; the angle yOA is γ, so that the arc OA subtends 2γ at the centre of the largest circle.

* See H. Glauert, *Aerofoil and Airscrew Theory* (Cambridge, 1930), 58–93.

† See L. Bairstow, *Applied Aerodynamics* (Longmans, 1920), 122.

The region outside the z-figure has to be mapped on the upper half of a c-plane (the complex variable c will be identified later with the k^2 of the Jacobian elliptic functions). It is convenient, however, to use an intermediate variable w. The z-figure is a curvilinear triangle having angles 2π, π, π; by means of a bilinear transformation connecting w and z we transform the figure so that O is at infinity, A

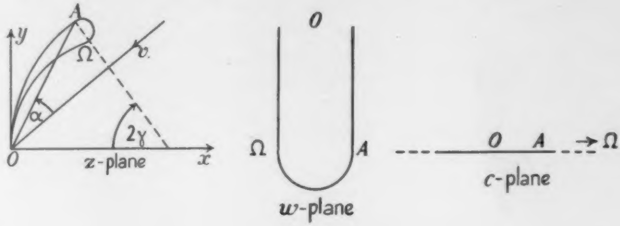

FIG. 1

is the origin, Ω is the point $w = -1$, and OA, ΩO are straight lines. Then OA, $A\Omega$, ΩO are lines in the standard elliptic modular configuration. The w-area to be mapped on the c-half-plane is to the left when the contour is described in the sense $OA\Omega$. The mapping is effected* by taking w to be the ratio of two appropriately chosen solutions of the hypergeometric equation

$$c(1-c)W''+3(1-c)W'-\tfrac{3}{4}W = 0,$$

where, in the ordinary notation, $\alpha = \frac{3}{2}$, $\beta = \frac{1}{2}$, $\gamma = 3$.

Now $$W_1 = \tfrac{1}{2}\int_0^1 t^{-\frac{1}{2}}(1-t)^{\frac{3}{2}}(1-ct)^{-\frac{3}{2}}\,dt,$$

and $$W_2 = \tfrac{3}{2}c'\int_0^1 t^{\frac{1}{2}}(1-t)^{-\frac{1}{2}}(1-c't)^{-\frac{3}{2}}\,dt \quad (c' = 1-c),$$

are independent solutions of this differential equation.

Write $t = \mathrm{sn}^2 u$ and $c = k^2$. With the usual notation for the complete elliptic integrals (of modulus k), the solution W_1 is

$$\int_0^K \frac{\mathrm{cn}^4 u}{\mathrm{dn}^2 u}\,du, \qquad \text{i.e. } \frac{1}{k^2}\int_0^K \mathrm{cn}^2 u\left(1-\frac{k'^2}{\mathrm{dn}^2 u}\right)du,$$

where k' is the complementary modulus.

* H. A. Schwarz, *J. für Math.* 75 (1872), 292–335. For an account see A. R. Forsyth, *Theory of Functions* (3rd edition, 1918), ch. xx.

But
$$\frac{d}{du}\left(\frac{\operatorname{sn} u \operatorname{cn} u}{\operatorname{dn} u}\right) = \frac{\operatorname{cn}^2 u}{\operatorname{dn}^2 u} - \operatorname{sn}^2 u;$$

therefore
$$\int_0^K \frac{\operatorname{cn}^2 u}{\operatorname{dn}^2 u}\, du = \int_0^K \operatorname{sn}^2 u\, du,$$

and hence
$$W_1 = \frac{1}{k^4}\int_0^K \{(1+k'^2)\operatorname{dn}^2 u - 2k'^2\}\, du = \frac{1}{c^2}\{(1+c')E - 2c'K\}.$$

To express W_2 in a similar form it is convenient to drop the accent on the c' so as to avoid any confusion in dealing with the Jacobian elliptic functions with the complementary modulus k'. Then, with the same substitutions as before,

$$\tfrac{3}{2}c \int_0^1 t^{\frac{1}{2}}(1-t)^{-\frac{1}{2}}(1-ct)^{-\frac{5}{2}}\, dt = 3c \int_0^K \frac{\operatorname{sn}^2 u}{\operatorname{dn}^4 u}\, du.$$

But
$$\frac{d}{du}\left(\frac{\operatorname{sn} u \operatorname{cn} u}{\operatorname{dn}^3 u}\right) = \frac{\operatorname{cn}^2 u + 2\operatorname{sn}^2 u}{\operatorname{dn}^2 u} - 3k'^2 \frac{\operatorname{sn}^2 u}{\operatorname{dn}^4 u};$$

hence
$$3c \int_0^K \frac{\operatorname{sn}^2 u}{\operatorname{dn}^4 u}\, du = \frac{c}{c'} \int_0^K \frac{\operatorname{cn}^2 u + 2\operatorname{sn}^2 u}{\operatorname{dn}^2 u}\, du$$

$$= \frac{c}{c'^2} \int_0^K \frac{c'\operatorname{cn}^2 u + 2(\operatorname{dn}^2 u - \operatorname{cn}^2 u)}{\operatorname{dn}^2 u}\, du$$

$$= \frac{1}{c'^2}\{(1+c)E - c'K\}.$$

Thus, restoring the accent for c', we get

$$W_2 = \frac{1}{c^2}\{(1+c')E' - cK'\},$$

E' and K' being complete elliptic integrals to modulus k'.

The function which maps the region external to the w-figure upon the upper half of the c-plane is therefore given by

$$w = \frac{A_1 W_1 + A_2 W_2}{B_1 W_1 + B_2 W_2},$$

where A_1, A_2, B_1, B_2 are constants to be determined.

As $c \to 0$ through positive real values, $E' \to 1$, and $K' \sim \log(4/k)$, that is, $cK' \to 0$ like $c \log c$. Also $W_1 = \frac{3}{16}\pi c^2 + O(c^3)$. Hence, if $w \to \infty$

as $c \to 0$, then $B_2 = 0$. Similarly, if $w \to 0$ as $c' \to 0$ through positive real values, $A_1 = 0$. Moreover, as $c \to 0$, $w \to \infty$ in the direction $+i\infty$. Therefore

$$w = iC\frac{(1+c')E'-cK'}{(1+c')E-2c'K},$$

where C is a real constant.

In the neighbourhood of $c = 0$, E' and K' are many-valued. In fact*

$$\pi E' = (K-E)\log\frac{16}{c}+(1-\tfrac{1}{4}c-\tfrac{13}{64}c^2-...),$$

$$\pi K' = K\log\frac{16}{c}-(\tfrac{1}{4}c+\tfrac{21}{128}c^2+...),$$

and therefore

$$w = \frac{iC}{\pi}\log c + \text{a function one-valued near } c = 0.$$

Let a c-point describe a small semicircle about $c = 0$, from $c = \epsilon$ to $c = \epsilon\exp(i\pi)$; the value of w is increased by $-C$. If $C = +1$, then w is increased by -1; this brings the line $O\Omega$ into its proper position in the elliptic modular configuration. Hence

$$w = i\frac{(1+c')E'-cK'}{(1+c')E-2c'K}. \tag{1.1}$$

Inserting in (1.1) the above expressions for E' and K', we get

$$w = \frac{i}{\pi}\log c+i\frac{32}{3\pi}\left(\frac{1}{c^2}-\frac{1}{c}\right)+\text{constant}+O(c), \tag{1.2}$$

in the neighbourhood of $c = 0$.

The transformation $$w-w_0 = -\frac{a}{z}, \tag{1.3}$$

where w_0 is the point in the w-plane which corresponds to c_0 in the c-plane, maps the space outside the z-figure into the space outside the w-figure. The four real parameters in the complex numbers w_0 and a suffice to ensure that the z-figure has the y-axis as the common tangent at O, and to determine the radii of the circles composing the figure.

2. Now consider the fluid motion in the c-plane whose velocity-potential ϕ and stream-function ψ are given by

$$\chi \equiv \phi+i\psi = \frac{V'\exp(i\delta)}{c-c_0}+\frac{V'\exp(-i\delta)}{c-\bar{c}_0}-\frac{i\kappa}{2\pi}\log\frac{c-c_0}{c-\bar{c}_0}, \tag{2.1}$$

* See, for example, A. Cayley, *Elliptic Functions* (2nd edition, 1895), 54.

where V', δ, κ (the circulation) are real constants, and $\bar{c}_0$ is conjugate to c_0. The line $OA\Omega$ is obviously a stream-line.

In the neighbourhood of c_0 let $w-w_0$ be expanded in a power series; so that

$$-\frac{a}{z} = w-w_0 = -\alpha_1(c-c_0)-\alpha_2(c-c_0)^2-\alpha_3(c-c_0)^3-\ldots, \tag{2.2}$$

where α_1, α_2, α_3,... are complex constants. We then find that the principal part of χ (*qua* function of z) at infinity in the z-plane is

$$\frac{\alpha_1}{a}V'z\exp(i\delta)+\frac{i\kappa}{2\pi}\log z.$$

Hence ϕ and ψ are also the velocity-potential and stream-function for the fluid motion past the z-figure, with a circulation about the figure.

Moreover, we get, from (2.2), the following expansions valid in the neighbourhood of c_0:

$$\left.\begin{aligned} c-c_0 &= \frac{a}{\alpha_1 z}\left(1-\frac{p_1}{z}+\frac{p_2}{z^2}+\ldots\right) \\ (c-c_0)^{-1} &= \frac{\alpha_1 z}{a}\left(1+\frac{p_1}{z}+\frac{p_1^2-p_2}{z^2}+\ldots\right) \\ (c-\bar{c}_0)^{-1} &= \frac{1}{\beta}\left(1-\frac{q_1}{z}+\ldots\right) \end{aligned}\right\} \tag{2.3}$$

where

$$p_1 = \frac{a\alpha_2}{\alpha_1^2}, \qquad p_2 = \frac{a^2(2\alpha_2^2-\alpha_1\alpha_3)}{\alpha_1^4}, \qquad q_1 = \frac{a}{\alpha_1\beta}, \qquad \beta = c_0-\bar{c}_0.$$

Now $-\partial\phi/\partial x$ and $-\partial\phi/\partial y$ are the components of velocity of the fluid in the z-plane. Also

$$\frac{d\chi}{dz} = \frac{\partial}{\partial x}(\phi+i\psi) = \frac{\partial\phi}{\partial x}-i\frac{\partial\phi}{\partial y}.$$

The component velocities at infinity in the z-plane being

$$-v\sin(\alpha+\gamma), \qquad -v\cos(\alpha+\gamma)$$

respectively [so that the direction of the flow at infinity makes an angle α with the straight line OA (Fig. 1)], it follows that

$$-iv\exp(i\,\overline{\alpha+\gamma}) = \left(\frac{d\chi}{dz}\right)_{z\to\infty} = \left(\frac{d\chi}{dc}\frac{dc}{dz}\right)_{c\to c_0}.$$

Hence, from (2.1) and (2.3),

$$V'\exp(i\delta) = -i\frac{a}{\alpha_1}v\exp(i\,\overline{\alpha+\gamma}). \tag{2.4}$$

Further, the value of the arbitrary circulation κ will be adjusted so that the Joukowski condition for smooth flow at the sharp trailing edge, i.e. $(d\chi/dc)_{c=0} = 0$, is satisfied. The corresponding value of κ is given by

$$\frac{i\kappa}{2\pi}\beta c_0 \bar{c}_0 = V'\{c_0^2 \exp(-i\delta) + \bar{c}_0^2 \exp(i\delta)\}, \tag{2.5}$$

and κ is, of course, real.

Apart from the obvious difficulty arising from the determination of the precise values of c_0, V', δ, and κ, in terms of the known constants, the complex velocity-potential for the flow past the z-figure has now been determined.

An expression for the resultant force per unit breadth of the aerofoil whose section is the z-figure may be obtained by applying a general theorem.* This resultant force may be reduced to a force (X, Y) acting at O and a couple G, given by

$$iX + Y = -\tfrac{1}{2}\rho \int \left(\frac{d\chi}{dz}\right)^2 dz,$$

$$G = -\Re\left[\tfrac{1}{2}\rho \int \left(\frac{d\chi}{dz}\right)^2 z\, dz\right],$$

where ρ is the density of the fluid, and the integrations in this case may be taken round a circle with centre O and large radius. It therefore becomes necessary to expand $d\chi/dz$ in a form convenient for evaluating the contour integrals.

Using (2.1) and (2.3) we find that

$$\frac{d\chi}{dz} = \frac{\alpha_1}{a} V' \exp(i\delta) + \frac{i\kappa}{2\pi}\frac{1}{z} + \left\{(p_2 - p_1^2)\frac{\alpha_1}{a^2} V' \exp(i\delta) + \right.$$
$$\left. + \frac{a}{\alpha_1 \beta^2} V' \exp(-i\delta) - \frac{i\kappa}{2\pi}\left(p_1 + \frac{a}{\alpha_1 \beta}\right)\right\}\frac{1}{z^2} + O\left(\frac{1}{z^3}\right);$$

that is,

$$\frac{d\chi}{dz} = -iv \exp(i\,\overline{\alpha+\gamma}) + \frac{i\kappa}{2\pi}\frac{1}{z} + \left\{-iv \exp(i\,\overline{\alpha+\gamma})\frac{a^2}{\alpha_1^4}(\alpha_2^2 - \alpha_1\alpha_3) - \right.$$
$$\left. - i\frac{a^2}{\alpha_1^2 \beta^2} v \exp(i\,\overline{\alpha+\gamma-2\delta}) - \frac{i\kappa}{2\pi}\frac{a}{\alpha_1}\left(\frac{\alpha_2}{\alpha_1} + \frac{1}{\beta}\right)\right\}\frac{1}{z^2} + O\left(\frac{1}{z^3}\right).$$

* H. Blasius, *Zeits. für Math.* 58 (1910), 90–110; H. Lamb, *Hydrodynamics* (6th edition, 1932), 91.

Since only the terms in z^{-1} in the integrands contribute anything to the integrals, we have

$$iX+Y = \frac{\alpha_1}{a}\kappa\rho V'\exp(i\delta),$$

and therefore

$$X = -\kappa\rho v\cos(\alpha+\gamma), \qquad Y = \kappa\rho v\sin(\alpha+\gamma).$$

These two components are together equivalent to a single force of magnitude $\kappa\rho v$ and direction at right angles to that of the stream at infinity. This is a mere verification of the general theorem of Kutta and Joukowski.

Again, the purely imaginary terms of the integral being omitted, G is the real part of

$$2\pi\rho v\left\{iv\exp(i\,\overline{2\alpha+2\gamma})\frac{a^2}{\alpha_1^4}(\alpha_2^2-\alpha_1\alpha_3)+\frac{i\kappa a}{2\pi\alpha_1}\left(\frac{\alpha_2}{\alpha_1}+\frac{1}{\beta}\right)\exp(i\,\overline{\alpha+\gamma})\right\}.$$

3. The calculation of the resultant force per unit breadth of the aerofoil is simplified for practical purposes by assuming that the radius of the leading edge is small in comparison with the length of the chord. On this assumption the values of the constants occurring in the above expressions for X, Y, and G may be determined.

Let r be the radius of the leading edge, R and R_1 $(R > R_1)$ the radii of the circles which touch at O, and h the length of the chord (here taken to be the straight line OA). The approximate values of the constants will be determined in terms of r, R, h.

The points O, A, Ω in the z-plane are respectively

$$0, \qquad \frac{a}{w_0}, \qquad \frac{a}{w_0+1}.$$

The geometry of the z-figure therefore gives

$$a = \frac{2RR_1}{R-R_1}, \qquad w_0 = -i\frac{R_1}{R-R_1}\operatorname{cosec}\gamma\exp(i\gamma).$$

But $$\sin\gamma = \frac{h}{2R}, \quad\text{and}\quad R_1 = \frac{R(R-r)}{R+r\cot^2\gamma}.$$

Hence $$a = \tfrac{1}{2}h^2\left(\frac{1}{r}-\frac{1}{R}\right) \sim \frac{h^2}{2r},$$

and $$w_0 = -i\tfrac{1}{2}h\left(\frac{1}{r}-\frac{1}{R}\right)\exp(i\gamma) \sim -i\frac{h}{2r}\exp(i\gamma).$$

Evidently c_0 is a point very near to the origin in the c-plane. In fact, from (1.2), we have

$$c_0^2 = i\frac{32}{3\pi w_0},$$

approximately, so that

$$c_0 = b\exp(i\,\overline{\tfrac{1}{2}\pi-\tfrac{1}{2}\gamma}),$$

where b is written for $(64r/3\pi h)^{\frac{1}{2}}$.

Again, from (2.2),

$$\alpha_1 = -\frac{dw}{dc}, \qquad 2\alpha_2 = -\frac{d^2w}{dc^2}, \qquad 6\alpha_3 = -\frac{d^3w}{dc^3},$$

the derivatives to be evaluated at $c = c_0$. Hence, from (1.2),

$$\alpha_1 = i\frac{32}{3\pi}\frac{2}{c_0^3}, \qquad \alpha_2 = -i\frac{32}{3\pi}\frac{3}{c_0^4}, \qquad \alpha_3 = i\frac{32}{3\pi}\frac{4}{c_0^5},$$

approximately.

Now

$$V'\exp(i\delta) = -i\frac{a}{\alpha_1}v\exp(i\,\overline{\alpha+\gamma}) = \tfrac{1}{2}hbv\exp(i\,\overline{\tfrac{1}{2}\pi+\alpha-\tfrac{1}{2}\gamma}),$$

and therefore $\qquad V' = \tfrac{1}{2}hbv, \qquad \delta = \tfrac{1}{2}\pi+\alpha-\tfrac{1}{2}\gamma.$

Similarly, the expression (2.5) for the circulation is, to this degree of approximation, equivalent to

$$\kappa = -\frac{2\pi V'}{b}\sec\tfrac{1}{2}\gamma\cos(\gamma+\delta) = \pi hv\sec\tfrac{1}{2}\gamma\sin(\alpha+\tfrac{1}{2}\gamma).$$

The force at O is therefore of magnitude

$$\pi\rho hv^2\sec\tfrac{1}{2}\gamma\sin(\alpha+\tfrac{1}{2}\gamma).$$

Also, the approximate values of

$$i\frac{a^2}{\alpha_1^4}(\alpha_2^2-\alpha_1\alpha_3) \quad\text{and}\quad \frac{i\kappa}{2\pi}\frac{a}{\alpha_1}\left(\frac{\alpha_2}{\alpha_1}+\frac{1}{\beta}\right)\exp(i\,\overline{\alpha+\gamma})$$

are respectively $\qquad -\tfrac{1}{16}h^2\exp(i\,\overline{\tfrac{1}{2}\pi-2\gamma})$

and $\qquad \tfrac{1}{8}h^2v\sec\tfrac{1}{2}\gamma\sin(\alpha+\tfrac{1}{2}\gamma)\{3\exp(i\alpha)-\sec\tfrac{1}{2}\gamma\exp(-i\tfrac{1}{2}\gamma)\}.$

The couple G is therefore the real part of

$$-2\pi\rho h^2v^2[\tfrac{1}{16}\exp(i\,\overline{\tfrac{1}{2}\pi+2\alpha})-\tfrac{1}{8}\sec\tfrac{1}{2}\gamma\sin(\alpha+\tfrac{1}{2}\gamma)\times$$
$$\times\{3\exp(i\alpha)-\sec\tfrac{1}{2}\gamma\exp(i\,\overline{\alpha-\tfrac{1}{2}\gamma})\}],$$

i.e.

$$G = \tfrac{1}{8}\pi\rho h^2v^2\{3\sec\tfrac{1}{2}\gamma\sin(2\alpha+\tfrac{1}{2}\gamma)-\tan^2\tfrac{1}{2}\gamma\sin 2\alpha+\tan\tfrac{1}{2}\gamma\}.$$

The determination of the resultant single force is now a simple matter. Let the line of action of this resultant force meet the chord at a point P. Then the ratio $\mu = AP/AO$ is given by

$$\mu = 1 - \frac{G}{hF}\sec\alpha,$$

F being the magnitude of the force at O.

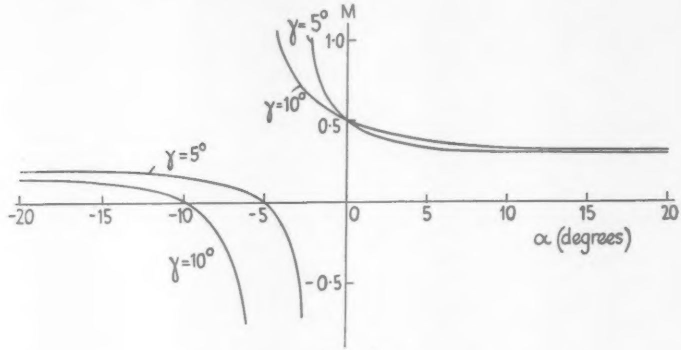

FIG. 2

The two curves (Fig. 2) illustrate the variation of μ with α for the values 5° and 10° of γ. The range of values of α is in both cases from $-20°$ to $20°$.

It is evident that the value of μ in both these cases remains nearly constant and equal to about one-third, for values of α between 5° and 20°.

THE DIFFERENTIAL EQUATIONS OF APPELL'S FUNCTION F_4

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1. Some years ago Watson,* using contour integrals, obtained certain relations between products of ordinary hypergeometric functions and an Appell Function F_4 of specialized type.† Later Bailey‡ obtained a more symmetrical relation of the same kind by direct transformation of the series involved. The purposes of the present note are to exhibit these relations as a necessary consequence of the form of the differential equations satisfied by F_4 and somewhat to extend the results previously known. The formulae obtained are of course only valid if the parameters of the functions involved avoid certain exceptional values: they will, for instance, require modification if the parameters or their differences are integral.§

2. The function F_4 is defined by the equality

$$F_4(\alpha,\beta;\gamma,\gamma';x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\alpha,m+n)(\beta,m+n)}{(\gamma,m)(\gamma',n)}\frac{x^m y^n}{m!\,n!}, \tag{1}$$

where $(\gamma,m) \equiv \gamma(\gamma+1)...(\gamma+m-1)$, etc. The series is convergent when $\sqrt{|x|}+\sqrt{|y|} < 1$, but the function may be analytically continued over a wider field by a Barnes' integral of appropriate type. We note that

$$\left.\begin{aligned} F_4(\alpha,\beta;\gamma,\gamma';x,0) &= F(\alpha,\beta;\gamma;x) \\ F_4(\alpha,\beta;\gamma,\gamma';0,y) &= F(\alpha,\beta;\gamma';y) \end{aligned}\right\}. \tag{2}$$

Throughout the discussion it is assumed that the parameters satisfy the equality

$$\gamma+\gamma' = \alpha+\beta+1. \tag{3}$$

The partial differential equations satisfied by F_4 may then be written||

$$\left.\begin{aligned} x(1-x)r-2xys-y^2t+\gamma p-(\gamma+\gamma')(xp+yq) &= \alpha\beta z \\ y(1-y)t-2xys-x^2r+\gamma' q-(\gamma+\gamma')(xp+yq) &= \alpha\beta z \end{aligned}\right\}. \tag{4}$$

* Watson (**1**).

† For a full discussion of the main properties of F_4 see (**2**).

‡ Bailey (**3**). § Cf. (**1**), 190. || (**2**) 44.

In future we denote by F_{41}, the function defined by (1) and set*

$$\left.\begin{aligned} F_{42} &= x^{1-\gamma}F_4(\gamma'-\alpha,\gamma'-\beta;\,2-\gamma,\gamma';\,x,y)\\ F_{43} &= y^{1-\gamma'}F_4(\gamma-\alpha,\gamma-\beta;\,\gamma,2-\gamma';\,x,y)\\ F_{44} &= x^{1-\gamma}y^{1-\gamma'}F_4(1-\alpha,1-\beta;\,2-\gamma,2-\gamma';\,x,y)\end{aligned}\right\}. \tag{5}$$

The functions F_{4r} $(r = 1,2,3,4)$ are linearly independent solutions of the system (4) and any solution of that system is a linear combination of them.

The results obtained by Watson and by Bailey suggest that it may be profitable to transform (4) by the substitution

$$x = XY, \qquad y = X'Y',$$

where

$$X' = 1-X, \qquad Y' = 1-Y.$$

Since X, Y are the roots of a quadratic equation

$$t^2-(x-y+1)t+x = 0,$$

we define X as that root of the equation which tends to zero with x.

More precisely, we may choose positive constants R_1, R_2 such that $\sqrt{R_1}+\sqrt{R_2} = 1$ and confine X, Y respectively to the regions in their planes defined by

$$|1-X| < 1, \quad |X| < R_1; \qquad |Y| < 1, \quad |1-Y| < R_2.$$

The condition $\sqrt{|x|}+\sqrt{|y|} < 1$ will then be satisfied and the series for the F_{4r} will converge.

If the dependent variable be denoted indifferently by z or Z and $P = \partial Z/\partial X$, $R = \partial^2 Z/\partial X^2$, etc., the formulae of transformation are

$$\left.\begin{aligned} &x = XY, \qquad y = X'Y' = (1-X)(1-Y)\\ &(Y-X)p = X'P-Y'Q, \qquad (Y-X)q = XP-YQ\\ &(Y-X)^3r = (Y-X)(RX'^2-2SX'Y'+TY'^2)+\\ &\qquad\qquad +2X'Y'(P-Q)\\ &(Y-X)^3t = (Y-X)(RX^2-2SXY+TY^2)+\\ &\qquad\qquad +2XY(P-Q)\\ &(Y-X)^3s = (Y-X)(RXX'-SXY'-SX'Y+TYY')+\\ &\qquad\qquad +(XY'+X'Y)(P-Q)\end{aligned}\right\}. \tag{6}$$

On transformation the system (4) becomes

$$X'[XX'R+(\gamma X'-\gamma'X)P-\alpha\beta Z] = Y'[YY'T+(\gamma Y'-\gamma'Y)Q-\alpha\beta Z]$$

$$X[XX'R+(\gamma X'-\gamma'X)P-\alpha\beta Z] = Y[YY'T+(\gamma Y'-\gamma'Y)Q-\alpha\beta Z],$$

* (2) 51–2.

a system clearly equivalent to

$$\left.\begin{aligned} XX'R+(\gamma X'-\gamma' X)P-\alpha\beta Z &= 0 \\ YY'T+(\gamma Y'-\gamma' Y)Q-\alpha\beta Z &= 0 \end{aligned}\right\}, \tag{7}$$

where each equation involves one independent variable only and both are of standard hypergeometric form. We note that the fundamental transformations of the hypergeometric group

$$X = 1-X_1, \qquad Y = 1-Y_1,$$

$$X = \frac{1}{X_1}, \qquad Y = \frac{1}{Y_1},$$

correspond, respectively, to an interchange of the variables x, y in (4) and to the transformation

$$x = \frac{1}{x_1}, \qquad y = \frac{y_1}{x_1}.$$

3. Before entering on the discussion of the systems (4) and (7) it is convenient to introduce certain abbreviations.

Remembering the condition (3), write

$$\left.\begin{aligned} \phi_1(t) &= F(\alpha,\beta;\gamma;t) = (1-t)^{1-\gamma'}F(\gamma-\alpha,\gamma-\beta;\gamma;t) \\ \phi_2(t) &= F(\alpha,\beta;\gamma';1-t) = t^{1-\gamma}F(\gamma'-\alpha,\gamma'-\beta;\gamma';1-t) \\ \phi_3(t) &= t^{1-\gamma}F(\gamma'-\alpha,\gamma'-\beta;2-\gamma;t) \\ &= t^{1-\gamma}(1-t)^{1-\gamma'}F(1-\alpha,1-\beta;2-\gamma;t) \\ \phi_4(t) &= (1-t)^{1-\gamma'}F(\gamma-\alpha,\gamma-\beta;2-\gamma';1-t) \\ &= t^{1-\gamma}(1-t)^{1-\gamma'}F(1-\alpha,1-\beta;2-\gamma';1-t) \end{aligned}\right\}, \tag{8}$$

where F is the ordinary hypergeometric function. Further, let

$$\left.\begin{aligned} \frac{\Gamma(\gamma)\Gamma(1-\gamma')}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)} &= \lambda, & \frac{\Gamma(\gamma')\Gamma(1-\gamma)}{\Gamma(\gamma'-\alpha)\Gamma(\gamma'-\beta)} &= \lambda' \\ \frac{\Gamma(\gamma)\Gamma(\gamma'-1)}{\Gamma(\alpha)\Gamma(\beta)} &= \mu, & \frac{\Gamma(\gamma')\Gamma(\gamma-1)}{\Gamma(\alpha)\Gamma(\beta)} &= \mu' \\ \frac{\Gamma(1-\gamma)\Gamma(2-\gamma')}{\Gamma(1-\alpha)\Gamma(1-\beta)} &= \rho, & \frac{\Gamma(1-\gamma')\Gamma(2-\gamma)}{\Gamma(1-\alpha)\Gamma(1-\beta)} &= \rho' \end{aligned}\right\}. \tag{9}$$

We know that the functions ϕ satisfy the standard hypergeometric equation and that there exist between them the relations*

$$\left.\begin{aligned} \phi_1 &= \lambda\phi_2+\mu\phi_4 \\ \phi_2 &= \lambda'\phi_1+\mu'\phi_3 \end{aligned}\right\}. \tag{10}$$

* Gauss (4) 213 (86).

We may also verify without difficulty that

$$\lambda\lambda'-1 = -\mu\rho = -\mu'\rho' = \frac{\sin\alpha\pi\sin\beta\pi}{\sin\gamma\pi\sin\gamma'\pi}.$$

We denote the common value of these expressions by Δ.

Any product $\phi_r(X)\phi_s(Y)$ is a solution of (7) and, when expressed in terms of x, y, of (4). It may therefore be written as a linear combination of the functions F_{4r}. As Bailey has pointed out, the expression will be unambiguous only if $r = s$. Otherwise it will represent $\phi_r(X)\phi_s(Y)$ when X, Y are restricted to the regions previously assigned to them and $\phi_s(X)\phi_r(Y)$ when their regions of variation are interchanged.

Conversely, if we select any four mutually independent solutions of the system (7), say $\phi_r(X)\phi_s(Y)$ $(r, s = 1, 2)$, any solution of (4) may be written as a linear combination of them.

4. Two solutions of the system (4) are identical if they give the same values for z, p, q, s when $x = y = 0$. For the function F_{41} we then have

$$z = 1, \qquad p = \frac{\alpha\beta}{\gamma}, \qquad q = \frac{\alpha\beta}{\gamma'}, \qquad s = \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma\gamma'},$$

while, for $\phi_1(X)\phi_2(Y)$ at the point $X = 0$, $Y = 1$, we have

$$P = \frac{\alpha\beta}{\gamma}, \qquad Q = -\frac{\alpha\beta}{\gamma'}, \qquad S = -\frac{\alpha^2\beta^2}{\gamma\gamma'}.$$

But at this point, from (6),

$$p = P = \frac{\alpha\beta}{\gamma}, \qquad q = -Q = \frac{\alpha\beta}{\gamma'},$$

$$s = -S+P-Q = \frac{\alpha^2\beta^2+\alpha\beta(\gamma+\gamma')}{\gamma\gamma'} = \frac{\alpha\beta(\alpha+1)(\beta+1)}{\gamma\gamma'}.$$

Moreover, $\phi_1(0)\phi_2(1) = 1$.

We thus have $$F_{41} = \phi_1(X)\phi_2(Y), \tag{11}$$

the formula given by Bailey. If α is a negative integer, $\phi_2(Y)$ is a simple multiple of $\phi_1(Y)$, and we have a formula given by Watson.* If we make the necessary changes in the parameters in (11), we obtain the equivalent formulae:

$$\left.\begin{aligned} F_{42} &= \phi_3(X)\phi_2(Y) \\ F_{43} &= \phi_1(X)\phi_4(Y) \\ F_{44} &= \phi_3(X)\phi_4(Y) \end{aligned}\right\}. \tag{12}$$

* (1) 190.

From (11) and (12) we have immediately

$$F_{41}F_{44} = F_{42}F_{43}. \tag{13}$$

Equating in this coefficients of $x^m y^n$, we obtain the result,

If $\gamma+\gamma' = \alpha+\beta+1$, *then*

$$\sum_{r=0}^{m}\sum_{s=0}^{n}\frac{(1-\alpha,r+s)(1-\beta,r+s)(\alpha,m+n-r-s)(\beta,m+n-r-s)}{r!\,s!\,(m-r)!\,(n-s)!\,(2-\gamma,r)(2-\gamma',s)(\gamma,m-r)(\gamma',n-s)}$$
$$=\sum_{r=0}^{m}\sum_{s=0}^{n}\frac{(\gamma-\alpha,r+s)(\gamma-\beta,r+s)(\gamma'-\alpha,m+n-r-s)(\gamma'-\beta,m+n-r-s)}{r!\,s!(m-r)!(n-s)!(\gamma,r)(2-\gamma',s)(2-\gamma,m-r)(\gamma',n-s)}. \tag{14}$$

Again, from (11) and (12), we have

$$\lambda F_{41}+\mu F_{43} = \phi_1(X)[\lambda\phi_2(Y)+\mu\phi_4(Y)],$$
$$\lambda' F_{41}+\mu' F_{42} = \phi_2(Y)[\lambda'\phi_1(X)+\mu'\phi_3(X)],$$

i.e. by (10),

$$\left.\begin{aligned}\lambda F_{41}+\mu F_{43} &= \phi_1(X)\phi_1(Y)\\ \lambda' F_{41}+\mu' F_{42} &= \phi_2(X)\phi_2(Y)\end{aligned}\right\}, \tag{15}$$

the former of which is the final formula given by Watson.* By the proper changes of parameters we obtain two similar formulae,

$$\left.\begin{aligned}\rho' F_{42}-\frac{\lambda'\rho'}{\rho}F_{44} &= \phi_3(X)\phi_3(Y)\\ \rho F_{43}-\frac{\lambda\rho}{\rho'}F_{44} &= \phi_4(X)\phi_4(Y)\end{aligned}\right\}. \tag{16}$$

The equations (15) and (16) are not of course independent, since there is a linear relation between the four *symmetric* solutions $\phi_r(X)\phi_r(Y)$ of the system (7). It is readily obtained by considering the expression

$$\lambda'\phi_1(X)\phi_1(Y)-\phi_1(X)\phi_2(Y)-\phi_2(X)\phi_1(Y)+\lambda\phi_2(X)\phi_2(Y),$$

which may be written as

$$\frac{1}{\lambda}(\lambda\phi_2(X)-\phi_1(X))(\lambda\phi_2(Y)-\phi_1(Y))+\frac{\Delta}{\lambda}\phi_1(X)\phi_1(Y),$$

or again as

$$\frac{1}{\lambda'}(\lambda'\phi_1(X)-\phi_2(X))(\lambda'\phi_1(Y)-\phi_2(Y))+\frac{\Delta}{\lambda'}\phi_2(X)\phi_2(Y).$$

* (1) 194.

Thus, by using (10),

$$\Delta[\lambda'\phi_1(X)\phi_1(Y)-\lambda\phi_2(X)\phi_2(Y)]$$
$$= \lambda\mu'^2\phi_3(X)\phi_3(Y)-\lambda'\mu^2\phi_4(X)\phi_4(Y). \quad (17)$$

This relation holds over any region in which the hypergeometric functions concerned are defined and, setting $Y = X$, we have the relation between the squares of the ϕ_r.

Again, by employing (10), (11), (12),

$$\phi_2(X)\phi_1(Y) = [\lambda'\phi_1(X)+\mu'\phi_3(X)][\lambda\phi_2(X)+\mu\phi_4(X)]$$
$$= \lambda\lambda' F_{41}+\lambda\mu' F_{42}+\lambda'\mu F_{43}+\mu\mu' F_{44}, \quad (18)$$

whence

$$\phi_2(X)\phi_1(Y)-\phi_1(X)\phi_2(Y) = \Delta F_{41}+\lambda\mu' F_{42}+\lambda'\mu F_{43}+\mu\mu' F_{44}. \quad (19)$$

We obtain a formula appropriate to *all* circumstances by attaching the sign $\pm$ to the right-hand side of (19). Other similar formulae might be obtained, but it is sufficient to quote without proof the result

$$\mu\mu'[\phi_4(X)\phi_3(Y)-\phi_3(X)\phi_4(Y)] = \Delta[\phi_2(X)\phi_1(Y)-\phi_1(X)\phi_2(Y)]. \quad (20)$$

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NOTE ON A CONJECTURE OF ERDŐS

By J. GILLIS (*Belfast*)

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Introduction. In this paper I construct a continuous periodic function of x such that the curve $y = f(x)$ intersects any non-vertical straight line, if at all, in an infinite number of points. The existence of such a function was conjectured by Erdős. The construction of $f(x)$ is given in § 1, and the proof that $f(x)$ has the stated property in § 2.

Terminology. A function defined on some interval (a, b) will be called *piecewise linear* if

(i) it is continuous in (a, b), and

(ii) (a, b) can be divided into a finite number of intervals in each of which the function is linear.

A function defined on an infinite interval will be called piecewise linear if it is so in each finite interval. By the *graph* of a function $\psi(x)$ we mean, of course, the set of points with coordinates $\{x, \psi(x)\}$, where x ranges over the interval of definition of $\psi(x)$.

If $\psi(x)$ is piecewise linear and G is its graph, then the points of G where $\psi'(x)$ does not exist will be called the *vertices* of G and the portions of G between successive vertices will be called its *pieces*.

1. (*a*) Suppose that we are given a positive integer n and two points P, Q such that PQ is not parallel to either coordinate axis. Draw the rectangle $PLQM$ with PL, QM parallel to Ox and PM, QL to Oy. Divide PL and QM each into $2n+1$ equal intervals. Suppose that the points of division of PL are

$$P_0(= P),\ P_1,\ P_2,\ \ldots,\ P_{2n},\ P_{2n+1}(= L),$$

and that those of QM are

$$Q_0(= Q),\ Q_1,\ Q_2,\ \ldots,\ Q_{2n},\ Q_{2n+1}(= M).$$

Then let $\Gamma_{P,Q,n}$ denote the broken line

$$P_0\, Q_{2n}\, P_2\, Q_{2n-2} \cdots P_{2n-2}\, Q_2\, P_{2n}\, Q_0.$$

(*b*) Let $\{n_k\}$, $\{m_k\}$ $(k = 1,\ 2, \ldots)$ denote increasing sequences of positive integers and $\{\delta_k\}$ a decreasing sequence of positive real

numbers such that $\sum_{k=1}^{\infty} \delta_k$ is convergent. Some further restriction on the behaviour of these sequences will be necessary, but it will be simpler to state these as the need for them arises.

(*c*) If O is the origin and A the point with coordinates $(1, 1)$, let $f_1(x)$ be the function whose graph is Γ_{0,A,n_1}. Also let $\phi_1(x) = f_1(x)$. Suppose now that $f_k(x)$ has been defined on $(0, 1)$ and that it is piecewise linear. Suppose further that $f_k(0) = 0, f_k(1) = 1$. Let C_k denote the graph of $f_k(x)$, and take a chain of points

$$S_0(=O), S_1, S_2, ..., S_{n_k}(=A),$$

on C_k such that

(i) the chain includes the vertices of C_k, and

(ii) all the segments $S_r S_{r+1}$ $(r = 0, 1, 2,..., m_k-1)$ are of length less than δ_k.

Now modify C_k by replacing each segment $S_r S_{r+1}$ by the curve $\Gamma_{S_r,S_{r+1},n_{k+1}}$, and let $\phi_{k+1}(x)$ denote the function whose graph is obtained in this way. To obtain $f_{k+1}(x)$ we must modify $\phi_{k+1}(x)$ further. For each r of the range 1 to m_k such that S_r is not an extremum* of $f_k(x)$, let T'_r be the extremum of $\phi_{k+1}(x)$ nearest to S_r on the left and T''_r that on the right. Then there is a portion of the graph of $\phi_{k+1}(x)$ joining T'_r and T''_r. Replace this portion by $\Gamma_{T'_r,T''_r,n_{k+1}}$. Do this for every such r and the resulting graph defines $f_{k+1}(x)$. Obviously the induction may be continued indefinitely.

(*d*) Clearly $|f_{k+1}(x)-f_k(x)| < 2\delta_k$ and so, by the hypothesis on $\sum \delta_k$, $f_k(x)$ will tend uniformly to a limit as k tends to infinity. We denote this limit by $f(x)$ and its graph by C. Since, for each k, the function $f_k(x)$ is continuous, so also is $f(x)$.

(*e*) Now, for each k, define $f_k(x)$ in the interval $(1, 2)$ by

$$f_k(x) = f_k(2-x).$$

For other values of x let $f(x+2) = f(x)$. Then $f_k(x)$ is defined on $(-\infty, \infty)$ and so also is $f(x)$. I denote by G_k the graph of $f_k(x)$ as x goes from $-\infty$ to ∞ and by G that of $f(x)$.

2. *Proof.* The proof will best be understood in stages. Let l denote the line $y = mx+c$ where $0 < |m| < \infty$.

(*a*) Given any positive integer K, we can find $n_0 = n_0(K, l)$ such that G_{n_0} meets l in at least K points.

* i.e. maximum or minimum.

This follows immediately from the nature of the construction.

(b) If m, n are positive integers ($m > n$) and I_1, I_2, I_3 are three successive intersections of l with G_n, then there is an intersection of l with G_m lying between I_1 and I_3.

This, too, is immediately obvious from the construction.

(c) On the hypothesis of (b), the closed interval $[I_1, I_3]$ on l contains a point of G.

By (b), the interval $[I_1, I_3]$ contains a point of G_m for every $m > n$. Let P_m be the one nearest to I_1 (P_m may coincide with I_1). Then two cases may arise.

(i) There exists an infinite sequence $\{\mu_p\}$ of integers such that $P_{\mu_1} = P_{\mu_2} = P_{\mu_3} = \dots = P$ (say). Then, clearly, P lies on G and the result is true.

(ii) If the hypothesis in (i) is not satisfied, then the set of points P_m must have a limit point, P' (say). P' will lie on the curve G.

(d) If, for some n, G_n meets l in N points, then G meets l in at least $\frac{1}{3}N-l$ points.

For, let the intersections of l with G_n be $I_1, I_2, \dots, I_N$. The intervals $[I_1, I_3]$, $[I_4, I_6]$,... all contain points of G, by (c), and the number of these intervals is $[\frac{1}{3}N]$, i.e. is at least $\frac{1}{3}N-1$.

(e) Combining (a) and (d), we see that $f(x)$ has the stated property.

3. *Remarks.* (a) It is easily verified that, at every point, at least one of the extreme Dini derivatives of $f(x)$ must be infinite. For, if not, let ξ be a point where all four extreme derivatives are finite.

Then we can find a sector σ, with vertex at $[\xi, f(\xi)]$, not containing the vertical line $x = \xi$, and such that $y = f(x)$ lies inside σ for $|x-\xi| < \delta$, where $\delta > 0$. Let l' denote the line $y-f(\xi) = \alpha(x-\xi)$, where $|\alpha| > 1/\delta$ and such that l' is outside σ. Clearly, if $|x-\xi| > \delta$, then $f(\xi)+|\alpha(x-\xi)| > 1$, and so l' cannot meet C at any such value of x. On the other hand, if $|x-\xi| < \delta$, then $y = f(x)$ lies inside σ, and so there can again be no intersection. Hence l' meets G in only the one point ξ, and this contradicts the property proved for $f(x)$.

(b) If we consider the finite graph C, then it has the property that any non-vertical line λ meets it, if at all, in an infinite set of points. For it is clear from the proof in § 2 that, if λ cuts any C_n at all, it will meet C in an infinite number of points. On the other hand, if it meets C without cutting any C_n, then it must be a tangent to C. This is ruled out by the considerations above.

(*c*) It will be seen from the proof that the set of intersections of the curve with any non-vertical line is dense-in-itself. Since it is also closed, it is a perfect set, and, in particular, has cardinal $\mathfrak{C}$.

(*d*) Suppose that a continuous periodic function has the property that any non-vertical line meets its graph, if at all, in at least n points, and let K_n denote the class of all such functions. Clearly $K_n \supset K_{n+1} \supset K_\infty$. The remarks made in (*a*) about functions of K_∞ apply without change to functions of K_n ($n \geqslant 2$). I conjecture that $K_2 = K_\infty$.

ON TWO FUNCTIONAL EQUATIONS WHICH OCCUR IN THE THEORY OF CLOCK-GRADUATION

By M. CRUM (*Oxford*)

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1. LET A be a particle-observer possessing an arbitrarily graduated clock, in the sense used by Milne,* and Milne and Whitrow;† and let him make observations on a particle B by means of light-signals. If A sends out a light-signal at time t_1 by his clock, the signal will, after reflection by B, return to A at time t_2 by A's clock; and, if A makes a series of such observations, he will find empirically a function f such that $t_2 = f(t_1)$. A can assign coordinates t, r to each reflection-event at B, where $t = \frac{1}{2}\{f(t_1)+t_1\}$, $r = \frac{1}{2}c\{f(t_1)-t_1\}$.

A may now try to regraduate his clock so that B shall be in 'uniform relative motion'. If the regraduated clock reads time $T = \phi(t)$, the new coordinates of the events at B are

$$T = \tfrac{1}{2}[\phi\{f(t_1)\}+\phi(t_1)], \qquad R = \tfrac{1}{2}c[\phi\{f(t_1)\}-\phi(t_1)].$$

The motion has become uniform if $R = VT$, where V is a constant; if $(1+V/c)/(1-V/c) = \lambda$, this is the same as

$$\phi\{f(t)\} = \lambda\phi(t), \tag{1}$$

where t has been written for t_1.

If the functional equation (1) has any solution, it has infinitely many; for, if $\phi_1(t)$ is a solution, it may be verified that so also is

$$\phi_1(t)P\{\log\phi_1(t)\}$$

if P is any function with the period $\log\lambda$. But, if, for some t_0, $f(t_0) = t_0$, then there is at most one solution which has a given non-zero right-hand derivative at $t = t_0$ (except in the case when $f'(t_0) = 1$). More precisely the following is true.

THEOREM I. *Let $f(t)$ have a continuous positive derivative $f'(t)$ for $0 \leqslant t \leqslant a$; let $f(t) > t$ for $0 < t \leqslant a$; $f(0) = 0$, $f'(0) > 1$; and*

$$|f'(t)-f'(0)| \leqslant Ct^{\alpha} \quad (0 \leqslant t \leqslant a')$$

for some positive C, α, a'.

* E. A. Milne, *World-Structure*, Chap. 2 (1935).

† E. A. Milne and G. J. Whitrow, *Zeits. für Astrophys.* 15 (1938), 263.

Then, if $\lambda = f'(0)$, *there is one and only one function* $\phi(t)$ *satisfying* (1) *for* $0 \leqslant t \leqslant a$, *and having a given positive right-hand derivative* $\phi'(0)$ *at* $t = 0$. *This function* $\phi(t)$ *has a continuous positive derivative for* $0 \leqslant t \leqslant a$.

But, if $\lambda \neq f'(0)$, *there is no such solution.*

Here t_0 has, without loss of generality, been taken to be zero. The conditions imposed on $f(t)$, other than the Lipschitz condition, imply that B has a continuous velocity less than that of light;* and that B leaves A with positive velocity at time $t = 0$, and does not return to A in the time-interval considered.

It is not true that there is a unique solution $\phi(t)$ for which $\phi'(0) = 0$; for, if there is one such solution $\phi_1(t)$, then $C\phi_1(t)$ is another such solution; also $C\{\phi_1(t)\}^n$ $(n > 0)$ satisfies (1) but with a new value of λ.

Proof. It is convenient to consider the inverse function

$$g(t) = f^{-1}(t).$$

If $f(a) = b$, $f(a') = b'$, $g(t)$ has a continuous positive derivative for $0 \leqslant t \leqslant b$; $g(t) < t$ for $0 < t \leqslant b$; $g(0) = 0$, $g'(0) < 1$; and

$$|g'(t) - g'(0)| < Bt^{\alpha} \quad (0 < t \leqslant b').$$

From (1) we have, putting $g(t)$ for t,

$$\phi\{g(t)\} = \mu\phi(t), \tag{2}$$

where $\mu = 1/\lambda$.

Now suppose $\phi(t)$ is a solution of (2) having the properties required. Differentiating (2) and putting $t = 0$ we obtain (since $\phi'(0) > 0$)

$$g'(0) = \mu, \quad \text{i.e. } f'(0) = \lambda.$$

Putting $t = 0$ in (2) we now get $\phi(0) = 0$, since $\mu = g'(0) \neq 1$.

If $g^2(t)$ is written for $g\{g(t)\}$, and so on, (2) gives by induction

$$\phi\{g^n(t)\} = \mu^n\phi(t). \tag{3}$$

Now, since $g'(0)$ exists, $g(t)/t$ is continuous in the interval $0 \leqslant t \leqslant b$, and so not greater than an upper bound μ_1, which is attained and so is less than unity. Hence $g^n(t) \leqslant \mu_1^n t$, and $g^n(t) \to 0$ as $n \to \infty$, for $0 \leqslant t \leqslant b$. Therefore, if $\phi'(0)$ exists,

$$\lim_{n \to \infty} \frac{\phi\{g^n(t)\}}{g^n(t)}$$

* For the meaning of this phrase see Milne, *World-Structure*, § 26.

must exist and be equal to $\phi'(0)$. But from (3)

$$\phi(t) = \frac{\phi\{g^n(t)\}}{g^n(t)} \frac{g^n(t)}{\mu^n},$$

so that finally, if there is a function $\phi(t)$ satisfying the stated conditions,

$$\phi(t) = \phi'(0) \lim_{n\to\infty} \frac{g^n(t)}{\mu^n}. \tag{4}$$

If this limit exists, it in fact provides a solution of (2). For, from (4),

$$\phi\{g(t)\} = \phi'(0) \lim_{n\to\infty} \frac{g^{n+1}(t)}{\mu^n} = \mu\phi(t).$$

It remains to verify the existence of the limit in (4), and to show that $\phi(t)$ so defined has a continuous positive derivative. If $g^0(t) = t$,

$$\frac{d}{dt}\left\{\frac{g^n(t)}{\mu^n}\right\} = \prod_{r=0}^{n-1} \frac{g'\{g^r(t)\}}{\mu}. \tag{5}$$

If $g'(t) = \mu\{1+h(t)\}$, since $\mu = g'(0)$, the Lipschitz condition gives

$$|h(t)| \leqslant Bt^\alpha/\mu \quad (0 \leqslant t \leqslant b').$$

Since $g^n(t) \leqslant \mu_1^n t \leqslant \mu_1^n b$ for $0 \leqslant t \leqslant b$, $g^n(t) < b'$ for large enough n, and then

$$|h\{g^n(t)\}| \leqslant B(\mu_1^n b)^\alpha/\mu.$$

The product (5) thus converges uniformly when $n \to \infty$, and so

$$\lim_{n\to\infty} \frac{g^n(t)}{\mu^n} = \int_0^t \prod_{n=0}^{\infty} [1+h\{g^n(x)\}]\, dx,$$

which is a function with a continuous positive derivative, for

$$0 \leqslant t \leqslant b.$$

Now suppose the equation $g(t) = t$, which has the root $t = 0$, has a second root $t = t_1$; then, if $g'(t) > 0$ and $g(t) < t$ for $0 < t < t_1$, we may choose b in the above proof to be any number less than t_1; so that $\phi(t)$ is uniquely defined for $0 \leqslant t < t_1$, and increases steadily in this interval. Hence, when $t \to t_1$, $\phi(t)$ tends either to infinity or to a finite limit, ϕ_1 say; but, if $t \to t_1$, (2) gives $\phi_1 = \mu\phi_1$, and, since $\mu < 1$ and ϕ_1 cannot be zero, this is impossible. It follows that as $t \to t_1$, $\phi(t) \to \infty$.

This is to be expected from the kinematical origin of the problem; if $f(t) = t$ for $t = 0$ and $t = t_1$, the particles A and B coincide at times $t = 0$ and $t = t_1$. Provided these events both have finite

epochs by A's regraduated clock, A and B (who by this clock are in uniform relative motion) coincide at two distinct events, so that they must be permanently coincident; and this possibility has been excluded by the condition that $f'(0) > 1$.

2. We may suppose that both particles A and B possess 'clocks'. Milne and Whitrow have considered* the problem of the graduation (or regraduation) of B's clock so that it shall become 'congruent' with A's clock in a well-defined sense. They showed that, if a light-signal sent out from A at time t by A's clock is reflected by B at time $\chi(t)$ by B's clock and returns to A at time $f(t)$ by A's clock, then, if B's clock has been so graduated as to be congruent with A's, $\chi(t)$ is a solution of the functional equation

$$\chi\{\chi(t)\} = f(t).$$

Following a procedure given by Hardy,† Milne and Whitrow gave a construction for $\chi(t)$, given $f(t)$; the constructed solution was not unique, being arbitrary in a certain interval. Were this the whole of the matter it would be impossible to set up a unique standard of time at B which was a copy of a standard of time set up at a distant A. It was, however, pointed out by Milne in colloquiums at Oxford that the position is radically different if A and B possess an epoch of coincidence, and it is clear physically that the arbitrariness of B's congruent clock then disappears.

A result equivalent to this is obtained below, with certain subsidiary restrictions on the functions $f(t)$ and $\chi(t)$. We may without loss of generality suppose that the epoch t of coincidence is zero. If $\chi^{-1}(t)$ exists and is equal to $\theta(t)$ and if $f^{-1}(t) = g(t)$,

$$\theta\{\theta(t)\} = g(t). \tag{6}$$

THEOREM II. *If $f(t)$ satisfies the conditions of Theorem I, there is unique solution of* (6) *such that*

$$0 \leqslant \theta(t) \leqslant t,$$

and such that $\theta(t)$ has a right-hand derivative $\theta'(0)$ at $t = 0$.

Proof. If any such function θ exists, then since

$$\theta^{2n}(t) = \theta[\theta^{2(n-1)}\{\theta(t)\}], \qquad \theta^{2n}\{\theta(t)\} = \theta\{\theta^{2n}(t)\},$$

we have $$g^n(t) = \theta[g^{n-1}\{\theta(t)\}], \qquad g^n\{\theta(t)\} = \theta\{g^n(t)\}.$$

* Milne and Whitrow, *Zeits. für Astrophys.* 15 (1938), 268, § 13.

† G. H. Hardy, *Orders of Infinity* (1924), 31.

As in Theorem I, $g^{n-1}\{\theta(t)\} \to 0$ and $g^n(t) \to 0$ when $n \to \infty$, so that, if $\theta'(0)$ exists, it is equal to both limits

$$\lim_{n\to\infty} \frac{g^n(t)}{g^{n-1}\{\theta(t)\}}, \qquad \lim_{n\to\infty} \frac{g^n\{\theta(t)\}}{g^n(t)}.$$

We can therefore write

$$\theta'(0) = \lim_{n\to\infty} \frac{g^n(t)}{\{g'(0)\}^n} \lim_{n\to\infty} \frac{\{g'(0)\}^n}{g^{n-1}\{\theta(t)\}},$$

and
$$\theta'(0) = \lim_{n\to\infty} \frac{g^n\{\theta(t)\}}{\{g'(0)\}^n} \lim_{n\to\infty} \frac{\{g'(0)\}^n}{g^n(t)}.$$

Now, if $\mu = g'(0)$ and $\phi(t)$ is the function defined by (4) (with $\phi'(0) = 1$, say), these equations become

$$\theta'(0) = \mu \frac{\phi(t)}{\phi\{\theta(t)\}} = \frac{\phi\{\theta(t)\}}{\phi(t)},$$

which give
$$\theta'(0) = \mu^{\frac{1}{2}}$$
and
$$\phi\{\theta(t)\} = \mu^{\frac{1}{2}}\phi(t). \tag{7}$$

Since, from the proof of Theorem I, $\phi(t)$ is continuous and increases steadily, and

$$\phi\{g(t)\} = \mu\,\phi(t),$$

(7) defines a unique function $\theta(t)$ such that

$$g(t) < \theta(t) < t \tag{8}$$

for $0 < t \leqslant b$.

This function satisfies (6). For, by (7) and (2),

$$\phi\{\theta^2(t)\} = \mu^{\frac{1}{2}}\phi\{\theta(t)\} = \mu\,\phi(t) = \phi\{g(t)\};$$

and this implies (6) because ϕ steadily increases. Also $\theta(t)$, as defined by (7), has a continuous positive derivative given by

$$\theta'(t)\phi'\{\theta(t)\} = \mu^{\frac{1}{2}}\phi'(t).$$

A similar result holds for the equation

$$\theta^n(t) = g(t), \tag{6'}$$

of which a solution is given by

$$\phi\{\theta(t)\} = \mu^{1/n}\phi(t). \tag{7'}$$

It remains to examine the behaviour of $\theta(t)$ when t approaches a second root t_1 of the equation $g(t) = t$.

The inequalities (8) show that $\theta(t) \to t_1$ as $t \to t_1$. But $\theta'(t_1)$ does not in general exist; this will be shown by an example.

Let
$$g(t) = \tfrac{1}{4}t \qquad (0 \leqslant t \leqslant \tfrac{4}{5}-\epsilon),$$
$$g(t) = 1-4(1-t) \quad (\tfrac{4}{5}+\epsilon \leqslant t \leqslant 1),$$
and let $g(t)$ and $g'(t)$ be positive and continuous for $\frac{4}{5}-\epsilon \leqslant t \leqslant \frac{4}{5}+\epsilon$. Then the only solution of (6) which has a right-hand derivative at $t = 0$ is equal to $\frac{1}{2}t$ for $0 \leqslant t \leqslant \frac{4}{5}-\epsilon$. Now let
$$\gamma(t) = 1-g^{-1}(1-t), \qquad \omega(t) = 1-\theta^{-1}(1-t),$$
so that $\quad \gamma(t) = g(t) \quad (|t-\frac{4}{5}| > 4\epsilon) \quad$ and $\quad \omega^2(t) = \gamma(t)$;

it follows that the only $\omega(t)$ having a right-hand derivative at $t = 0$ is equal to $\frac{1}{2}t$ for $0 \leqslant t \leqslant \frac{4}{5}-4\epsilon$, i.e. that the only solution of (6) which has a left-hand derivative at $t = 1$ is equal to $1-2(1-t)$ for $\frac{3}{5}+2\epsilon \leqslant t \leqslant 1$. For $\epsilon < \frac{1}{15}$, the two solutions are unequal in the interval $\frac{3}{5}+2\epsilon \leqslant t \leqslant \frac{4}{5}-\epsilon$; so that no solution can have a derivative both at $t = 0$ and at $t = 1$.

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